

# Reliable Deniable Communication: Hiding Messages in Noise

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**Abstract**—A transmitter Alice may wish to *reliably* transmit a message to a receiver Bob over a binary symmetric channel (BSC), while simultaneously ensuring that her transmission is *deniable* from an eavesdropper Willie. That is, if Willie listening to Alice’s transmissions over a “significantly noisier” BSC than the one to Bob, he should be unable to estimate even whether Alice is transmitting. Even when Alice’s (potential) communication scheme is publicly known to Willie (with *no* common randomness between Alice and Bob), we prove that over  $n$  channel uses Alice can transmit a message of length  $\mathcal{O}(\sqrt{n})$  bits to Bob, deniably from Willie. We also prove information-theoretically order-optimality of our results.

## I. INTRODUCTION

Alice is in jail, and may wish to communicate reliably with Bob in the neighboring cell, over  $n$  uses of a noisy BSC (if she stays silent, the input to the channel is all zeroes). Unfortunately, the warden Willie is monitoring Alice (though his observations are significantly noisier, since his CCTV camera is low-quality).<sup>1</sup> Willie only wishes to detect Alice’s “transmission status” (*i.e.*, he only wants to know whether she’s talking or not, and doesn’t necessarily care *what* she’s saying). Hence Alice wishes to use a communication scheme that is “deniable from Willie”, *i.e.* Willie’s *best* estimate of Alice’s transmission status should be essentially statistically independent of his observations.

In this work we demonstrate:

- 1) *Deniability – outer bound on codeword weight*: If the binary code Alice uses to encode her message contains a substantial fraction of “high-weight codewords” (that is, have weight that is  $\omega(\sqrt{n})$  over  $n$  channel uses), then her communication scheme cannot be deniable. In particular, Willie can simply count the number of non-zero symbols he observes, and compare this number with a simple function of channel parameters, to estimate, fairly accurately, Alice’s transmission status. Hence, for deniability from Willie, Alice’s code should comprise mostly of “low-weight codewords”.
- 2) *Reliability/deniability – outer bound on throughput*: In our model the communication link from Alice to Bob is *also* a BSC (albeit a somewhat less noisy channel than the one from Alice to Willie). We use information-theoretic inequalities to demonstrate that for any code

that satisfies the outer bound on codeword weight required for deniability, if reliable decoding by Bob is *also* required, then a message of at most  $\mathcal{O}(\sqrt{n})$  bits can be transmitted by Alice over  $n$  channel uses.<sup>2</sup>

- 3) *Reliability/deniability – achievable scheme*: If Willie’s BSC is “sufficiently noisier” (in a precise sense that we quantify later) than Bob’s BSC, then we design a communication scheme (publicly known to all parties – Alice, Bob and Willie) such that:

- *Throughput*: It encodes a message with  $\mathcal{O}(\sqrt{n})$  bits in  $n$  channel uses (or, when the channel to Bob is noiseless (though the channel to Willie is still noisy)  $\mathcal{O}(\sqrt{n}) \log(b)$  bits).
- *Reliability*: It enables Bob to correctly reconstruct Alice’s message with high probability.
- *Deniability*: Willie’s best estimate of Alice’s transmission status is essentially statistically independent of his observations of his channel.

The first two results above are analogues (for the scenario of the BSCs considered in this work) of theorems in recent work that motivated our work (in particular, the corresponding results for Additive White Gaussian Noise (AWGN) channels proved in [1, 2]). The last, corresponding to construction of “reliable and deniable public codes” is the main result of this work.

In particular, we stress again that in our model (unlike the models in most prior work) everything that Bob knows *a priori* about Alice’s communication scheme, Willie *also* knows – there is no common randomness that is hidden from Willie that Alice and Bob can leverage. The only asymmetry between Bob’s and Willie’s estimation abilities arises from the fact that Willie’s observations of Alice’s (possible) transmissions are noisier than Bob’s. Hence the fact that we demonstrate the existence of *public* codes satisfying Result 3 above is a significant strengthening of the model in [1, 2], wherein in general common randomness is required, and consumed at a rate greater than the throughput of the reliable/deniability communication in the first place!

Also, in our model (and also in the model of [1, 2], but *not* in the vast majority of steganographic models), Alice’s default transmission if she has nothing to say, is nothing. This default silence of Alice makes it challenging to hide the

<sup>1</sup>If the channel from Alice to Willie is at least as good as the channel from Alice to Bob, then clearly no communication that is simultaneously reliable and deniable is possible, since Willie can use whatever decoding strategy Bob can use.

<sup>2</sup>Note that this implies that Alice’s rate decays to 0 asymptotically in  $n$ . Hence in this work we usually scale Alice’s “throughput” (the number of bits in her message) with respect to  $\sqrt{n}$ , to obtain a quantity we call the “relative throughput”.

fact that she is *not* silent when she actually has something to say. The only reason we are able to achieve a non-zero throughput is due to the fact that Willie’s observations of Alice’s potential transmissions are noisy (and in particular, significantly noisier than Bob’s). Hence the subtitle of this work – “hiding messages in noise”.

## II. RELATED WORK

The problem we consider is a variant of the classical *steganography* problem, but with important differences in the model that both make our results more “realistic” in some settings, and also technically more challenging to prove non-trivial results about.

### A. Steganography

The problem of steganography (broadly defined as “hiding an undetectable message in plain sight”) is rooted in antiquity – brief but colourful historical perspectives on a variety of stenographic models and methods (including various techniques used by Xerxes, Herodotus, Mary Queen of Scots, and Margaret Thatcher, and one which involves killing dogs) can be found in [3] and [4]. Information-theoretic models presenting “modern, formal characterizations” of steganography problems started appearing in the literature in the 1980’s and 1990’s – among many others the works of Simmons [5] (who formalized the “Prisoners’ problem”), and Cachin [6] and Maurer [7] (who drew the connection between steganography and another classical problem – *hypothesis testing*) come to mind. More recent and fairly comprehensive compendiums of results on the theory and practice of steganography can be found in the books [8] and [9].

However, the vast majority of steganographic models make at least one of the following assumptions (*none of which we make*):

- **(A1) Non-zero covertext/stegotext:** In almost all works in the literature, Alice has access to a length- $n$  sequence (the “covertext”) drawn from some distribution (this distribution, but not the actual value of the covertext, is known to Bob and Willie). The assumption is that after observing the covertext, Alice is allowed to transmit some (slightly) perturbed value of the covertext, called the “stegotext”, over the channel, and both Bob and Willie observe this stegotext (or some further “perturbed” version of it). The critical point here is that Alice’s *default* transmission (even if she has no hidden message to transmit to Bob) is usually non-zero, by this assumption. One example could be if Alice is allowed to upload photographs onto her website – this activity looks innocuous enough, and Willie might find it challenging to estimate whether or not Alice is hiding some message to Bob in those photographs. It is this “haystack” of covertexts/stegotexts that many steganographic algorithms leverage, to hide a “needle” of a hidden message in. A plethora of works characterize the “capacity” of various steganographic problems – see for instance [10–14]. An important exception to the non-zero covertext

assumption occurs in the work of Bash, Goeckel and Towsley [1, 2] – we discuss this work in depth below.

- **(A2) Shared secret key/common randomness:** Kerchoffs’ principle of cryptography [15] states, roughly, that “A cryptosystem should be secure even if everything about the system, except the key, is public knowledge”. This is just as true for the problem of steganography. However, a significant number of steganographic protocols violate this precept by requiring a key (that is often almost as large as the message being communicated) that is shared between Alice and Bob, and is kept secret from Willie, in advance of any communication. A variety of examples of such protocols can be found in, for example [9] or [10]. Such a key certainly helps considerably – it allows Alice and Bob to coordinate which of potentially many codes to use to communicate, and Willie is left in the dark regarding this choice. *The work that is closest in spirit to our work, that of [1, 2], differs critically with ours in this assumption – their protocol requires Alice to consume  $\Omega(n)$  bits of a secret key shared with Bob, for her to be able to communicate to Bob a hidden message with  $\mathcal{O}(\sqrt{n})$  bits.*

However, not all works make this assumption. Some exceptions to this assumption of a shared secret include works by [11, 14].<sup>3</sup>

- **(A3) Noiseless communication:** Some works consider a model wherein the communication channel between Alice and Bob is noiseless. This has some important consequences – in some such scenarios, the optimal throughput can sometimes be boosted by a multiplicative factor of  $\log n$  (for instance [9, Chapters 8 and 13]). However, in a variety of other works noise in the channel from Alice to Bob may have noise. In some models this may be due to an actively jamming warden (for instance [10]). In other models this may simply be random channel noise (for instance the work of [1, 2], and our work here).

### B. The Square Root Law

The “Square Root Law” (often abbreviated as SRL in the literature) can be perhaps characterized as an observation that in a variety of steganographic models, the throughput (the length of the message that Alice can communicate deniably and reliably with Bob) scales as  $\mathcal{O}(\sqrt{n})$  (here  $n$  is the number of “channel uses” that Alice has access to).

“Steganographic capacity is a loosely-defined concept, indicating the size of payload which may

<sup>3</sup>The scheme in the latter work has the property that the stegotext could have a large “distortion” when compared with the covertext (even though its statistics are identical, and hence the scheme is perfectly deniable). Various other works (for instance [10]) also impose “maximum distortion” criteria in addition to reliability and deniability. In our model this is a moot point. For one, our covertext is all-zero. For another, our codes have the property that Alice’s transmissions use “very low-weight” codewords, hence automatically satisfying most natural distortion criteria with respect to the all-zero codeword.

securely be embedded in a cover object using a particular embedding method. What constitutes “secure” embedding is a matter for debate, but we will argue that capacity should grow only as the square root of the cover size under a wide range of definitions of security.” – [16].

This observation seems to have some empirical support in the community of people implementing “real-world” steganographic protocols (for instance, see [17]). This “law” is heuristically justified via the following reasoning:

“Thanks to the Central Limit Theorem, the more covertext we give the warden, the better he will be able to estimate its statistics, and so the smaller the rate at which [the steganographer] will be able to tweak bits safely.” – [18]

“[T]he reference to the Central Limit Theorem... suggests that a square root relationship should be considered.” – [16]

Some recent work (for instance [12, 19]) has begun to theoretically justify this law under some (fairly restrictive) assumptions on the class of steganographic protocols. Nonetheless, results in this class should still be taken with a pinch of salt, since they do not offer a universally robust characterization for all models which may be of interest. For instance, in some works (for instance [9, Chapters 8 and 13]) the throughput scales as  $\mathcal{O}(\sqrt{n} \log(n))$ . More drastically, the works of [10] (which gives an information-theoretically optimal characterization of the rate-region of many variants of the steganography problem) and that of [11] (which design computationally efficient steganography protocols) both allow throughput that scales linearly in  $n$ , rather than  $\mathcal{O}(\sqrt{n})$  as would be indicated by the SRL. The major difference between the models of [10, 11], and those that satisfy the SRL, seems to lie in a disagreement as to what comprises “realistic” steganographic algorithms.

We note that in our setting (and also that of [1, 2]), our throughput does indeed provably scale as the square-root of the number of channel uses. However, the critical reason underlying this scaling is that we consider the scenario wherein the covertext is all-zero – Alice must “whisper very softly”, since she has no excuse if Willie hears something that cannot be explained by the noise on the channel to him.

### C. The work of Bash, Goeckel and Towsley [1, 2]

The results and techniques closest to those in this work (and indeed the starting-point of our investigations) are those of [1, 2]. However, there are important differences in the models.

- **Public codes vs. shared secret keys:** The critical difference between our model and that of [1, 2] (and the reason we state that our model is more “realistic”) is that in our setting there is no shared secret key between Alice and Bob that is hidden from Willie. Hence our codes are “public”. A setting wherein Alice’s consumption of secret keys happens significantly faster ( $\Omega(n)$ ) than her throughput ( $\mathcal{O}(\sqrt{n})$ ) to Bob (as in [1, 2]) is not sustainable.

The reason we are able to achieve such performance is due to a more intricate analysis of random binary codes than is carried out in [1, 2] – novel and intricate analysis of concentration inequalities was required.

- **Discrete vs. continuous channels:** In our work all channels are discrete (finite input and output alphabets) – in particular, for ease of presentation of our results we focus on the case wherein Alice’s transmissions pass through independent BSCs to get to Bob and Willie (though our results may be directly extended to larger classes of discrete channels). In contrast, the results of [1, 2] are for channels wherein the noise is AWGN. It is conceivable that our construction of public codes also carries over to the AWGN model of [1, 2], but significant extensions would be required to translate our techniques from the discrete world over to the continuous version.

## III. MODEL

### A. Notational Conventions

Calligraphic symbols such as  $\mathcal{C}$  denote sets. Boldface uppercase symbols such as  $\mathbf{X}$  denote random variables, boldface lower-case symbols such as  $\mathbf{x}$  denote particular instantiations of those random variables. Vectors are denoted by an arrow above a symbol, such as in  $\vec{x}$ . For notational convenience, in this work, unless otherwise specified, all vectors are of length  $n$ , where  $n$  corresponds to the *block-length* (number of channel uses). Probabilities of events are denoted with a subscript denoting the random variable(s) over which the probabilities are calculated. All logarithms in this work are binary, unless otherwise stated. The *Hamming weight* (number of non-zero entries) of a vector  $\vec{x}$  is denoted by  $wt(\vec{x})$ , and the *Hamming distance* between two vectors  $\vec{x}$  and  $\vec{y}$  of equal length (the number of corresponding entries in which  $\vec{x}$  and  $\vec{y}$  differ) is denoted by  $d(\vec{x}, \vec{y})$ . For any two numbers  $a$  and  $b$  in the interval  $[0, 1]$ , we use  $a * b$  to denote *binary convolution* of these two numbers, defined as  $a(1 - b) + b(1 - a)$  – this corresponds to the noise parameter of the BSC comprising of a BSC( $a$ ) followed by a BSC( $b$ ). As is standard in an information-theoretic context, the notation  $H(\cdot)$  corresponds to the (*binary*) *entropy function*,  $H(\cdot|\cdot)$  to *conditional entropy*,  $I(\cdot; \cdot)$  to *mutual information*, and  $D(\cdot||\cdot)$  to the *Kullback-Leibler divergence* between two distributions.

### B. Communication System

The transmitter Alice is connected via a binary-input binary-output broadcast medium to the receiver Bob and the warden Willie. The channel from Alice to Bob is a Binary Symmetric Channel with crossover probability  $p_b$  (henceforth denoted BSC( $p_b$ )). The channel from Alice to Willie is a BSC( $p_w$ ). By assumption, the noise on the two channels is independent,  $p_b < p_w$ , and all parties (Alice, Bob and Willie) know the channel parameters  $p_b$  and  $p_w$ .

Alice (potentially) wishes to communicate a *message*  $m$  uniformly at random from a set  $\{1, \dots, N\}$  to Bob – the symbol  $\mathbf{M}$  denotes the random variable corresponding to Alice’s message. For notational convenience we say that if

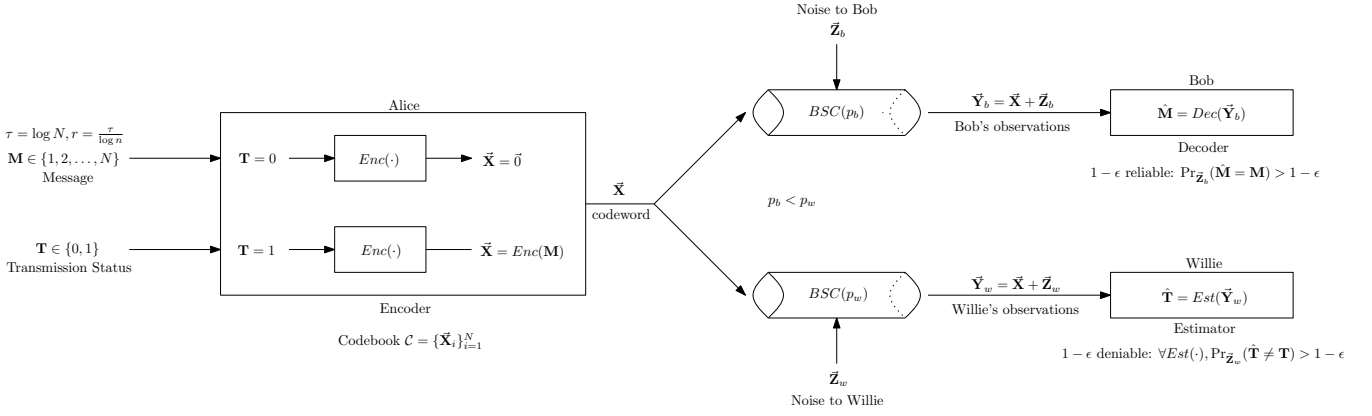


Fig. 1. Depending on her transmission status  $\mathbf{T}$ , Alice either broadcasts the all-zero vector  $\vec{0}$ , or encodes her messages  $\mathbf{M}$  into codewords  $\vec{\mathbf{X}} = \text{Enc}(\mathbf{M})$  from her codebook  $\mathcal{C}$ . The codebook has  $2^\tau = 2^{r\sqrt{n}}$  codewords, where the throughput  $\tau$  scales as  $\mathcal{O}(\sqrt{n})$ . Bob receives  $\vec{\mathbf{Y}}_b$  over a  $\text{BSC}(p_b)$ , and Willie receives a noisier version  $\vec{\mathbf{Y}}_w$  over a  $\text{BSC}(p_w)$ , where  $p_b < p_w$ . It is desired that the codebook  $\mathcal{C}$  be both reliable (i.e. Bob is able to decode  $\mathbf{M}$  as  $\hat{\mathbf{M}}$  with a “small” probability error) and deniable (i.e. Willie’s observations give him essentially no information about Alice’s transmission status.)

Alice does not wish to communicate with Bob, her message is 0. Equivalently, if Alice does have a message she wishes to communicate to Bob, then a certain arbitrary binary variable  $\mathbf{T}$  equals 1. Otherwise,  $\mathbf{T}$  equals 0. Only Alice knows the value of  $\mathbf{T}$  a priori.

Alice encodes each message  $m \in \{1, \dots, N\}$  into a length- $n$  binary codeword  $\vec{x}(m) = (\mathbf{x}(m, 1), \dots, \mathbf{x}(m, n))$  using an encoder  $\text{Enc}(\cdot) : \{0\} \cup \{1, \dots, N\} \rightarrow \{0, 1\}^n$ . To simplify notation, we often denote  $\vec{x}(m)$  and  $\vec{\mathbf{X}}(\mathbf{M})$  as  $\vec{x}$  and  $\vec{\mathbf{X}}$  respectively, wherever it does not cause confusion. The encoder is required to satisfy the condition that the 0 message is always encoded to the length- $n$  zero-vector  $\vec{0}$ . The set  $\{\vec{x}(1), \dots, \vec{x}(N)\}$  of possible non-zero codewords that are the outputs of Alice’s encoder is denoted by the codebook  $\mathcal{C}$ . The throughput  $\tau$  of Alice’s codebook is defined as  $\log N$ , and the relative throughput  $r$  of Alice’s codebook is defined as  $\log N / \sqrt{n}$ . We note that in this work, the throughput of the codes we consider typically scale with the block-length  $n$  as  $\mathcal{O}(\sqrt{n})$ . This corresponds to a “rate” that decays to zero as  $n$  increases without bound, rather than converging to a constant as is common in many other communication settings. Hence we deliberately consider the relative throughput rather than the rate of our codes.

Bob receives the length- $n$  binary vector  $\vec{\mathbf{Y}}_b$ . Here  $\vec{\mathbf{Y}}_b = \vec{\mathbf{X}} \oplus \vec{\mathbf{Z}}_b$ , where  $\vec{\mathbf{Z}}_b$  denotes the noise added by the  $\text{BSC}(p_b)$  channel between Alice and Bob. Bob uses his decoder  $\text{Dec}(\cdot) : \{0, 1\}^n \rightarrow \{0\} \cup \{1, \dots, N\}$  to generate his estimate of Alice’s message as  $\hat{\mathbf{M}} = \text{Dec}(\vec{\mathbf{Y}}_b)$ . Bob’s probability of decoding error when Alice is transmitting,  $P_{e, T=1}$ , is defined as  $\Pr_{\mathbf{M}, \vec{\mathbf{Z}}_b}(\hat{\mathbf{M}} = \vec{0} | \mathbf{T} = 1) + \Pr_{\mathbf{M}, \vec{\mathbf{Z}}_b}(\hat{\mathbf{M}} \neq \mathbf{M} | \mathbf{T} = 1)$ . Bob’s probability of decoding error when Alice is not transmitting,  $P_{e, T=0}$ , is defined as  $\Pr_{\vec{\mathbf{Z}}_b}(\hat{\mathbf{M}} \neq \vec{0} | \mathbf{T} = 0)$ . Bob’s overall probability of decoding error,  $P_e$ , is defined as

$P_{e, T=1} + P_{e, T=0}$ .<sup>4</sup> We say Alice’s codebook  $\mathcal{C}$  is  $(1 - \epsilon)$ -reliable if Bob’s probability of decoding error is less than  $\epsilon$ .

Willie knows a priori both  $\text{Enc}(\cdot)$  and  $\text{Dec}(\cdot)$  (and hence also  $\mathcal{C}$ ). Willie receives the length- $n$  binary vector  $\vec{\mathbf{Y}}_w$ . Here  $\vec{\mathbf{Y}}_w = \vec{\mathbf{X}} \oplus \vec{\mathbf{Z}}_w$ , where  $\vec{\mathbf{Z}}_w$  denotes the noise added by the  $\text{BSC}(p_w)$  channel between Alice and Willie. Willie uses his estimator  $\text{Est}_{\mathcal{C}}(\cdot) : \{0, 1\}^n \rightarrow \{0, 1\}$  to generate his estimate of Alice’s transmission status as  $\hat{\mathbf{T}} = \text{Est}_{\mathcal{C}}(\vec{\mathbf{Y}}_w)$ .

We use a hypothesis-testing metric to quantify the deniability of Alice’s codebook  $\mathcal{C}$ . Let the probability of false alarm  $\Pr_{\vec{\mathbf{Z}}_w}(\hat{\mathbf{T}} = 1 | \mathbf{T} = 0)$  be denoted by  $\alpha(\text{Est}_{\mathcal{C}}(\cdot))$ . Analogously, let the probability of missed detection  $\Pr_{\mathbf{M}, \vec{\mathbf{Z}}_w}(\hat{\mathbf{T}} = 0 | \mathbf{T} = 1)$  be denoted by  $\beta(\text{Est}_{\mathcal{C}}(\cdot))$  (and for a specific transmitted codeword  $m$ , we denote  $\Pr_{\vec{\mathbf{Z}}_w}(\hat{\mathbf{T}} = 0 | \mathbf{M} = m)$  by  $\beta^{(m)}(\text{Est}_{\mathcal{C}}(\cdot))$ ). The quantities  $\alpha(\text{Est}_{\mathcal{C}}(\cdot))$  and  $\beta(\text{Est}_{\mathcal{C}}(\cdot))$  denote respectively the probability that Willie guesses Alice is transmitting even if she is not, and the probability that Willie guesses Alice is not transmitting even though she actually is. We say Alice’s codebook  $\mathcal{C}$  is  $(1 - \epsilon)$ -deniable if there is no estimator  $\text{Est}_{\mathcal{C}}(\cdot)$  for Willie such that  $\alpha(\text{Est}_{\mathcal{C}}(\cdot)) + \beta(\text{Est}_{\mathcal{C}}(\cdot)) < 1 - \epsilon$ .<sup>5</sup>

For any block-length  $n$ , we say a corresponding codebook  $\mathcal{C}$  is simultaneously  $(1 - \epsilon)$ -reliable and  $(1 - \epsilon)$ -deniable if it simultaneously ensures that Bob’s probability of decoding error is at most  $\epsilon$ , and has deniability  $1 - \epsilon$ . A relative throughput  $r$  is said to be reliably and deniably achievable if for any  $\epsilon > 0$  there exists a sufficiently large block-length  $n$  and a corresponding codebook  $\mathcal{C}$  with relative throughput  $r$  over that block-length that is simultaneously  $(1 - \epsilon)$ -reliable and  $(1 - \epsilon)$ -deniable. The optimal relative throughput is the supremum of all reliably and deniably achievable relative

<sup>4</sup>To simplify matters we consider here only deterministic encoders and decoders, rather than the more general stochastic encoders and decoders, which are allowed to use private randomness not available to any other party. Examination of our techniques demonstrates that our results do not change substantially even if we allow stochastic encoding/decoding.

<sup>5</sup>Again, we restrict our attention to deterministic estimators – an averaging argument directly demonstrates that given any stochastic estimator, there exists a deterministic estimator with at least equivalent performance for Willie.

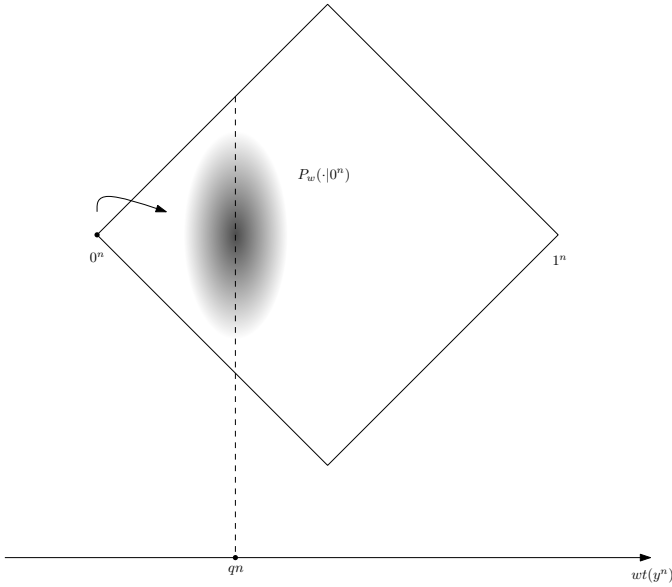


Fig. 2. The diamond at the top of this figure denotes the set (of size  $2^n$ ) of all possible  $\vec{y}_w$  that Willie may observe if Alice transmits nothing (her transmission status  $\mathbf{T} = 0$ ). In particular, the  $\vec{y}_w$  are arranged in a partial order, so that vectors with lower Hamming weight are to the left of vectors with higher Hamming weight. The curve at the bottom plots the probability distribution of observing  $\vec{y}_w$  of a particular weight. Since  $\vec{\mathbf{X}} = 0$ , the “typical”  $\vec{y}_w$  that Willie observes are of weight approximately  $p_w n$ . Also, the probability distribution on  $\vec{y}_w$  equals  $\Pr_{\vec{\mathbf{Z}}_b}(\vec{y}_w | \mathbf{T} = 0)$ .

throughputs.

#### IV. MAIN RESULTS/HIGH-LEVEL INTUITION

We now present our main results, and the intuition behind the corresponding proof techniques.

Theorem 1 below proves an outer bound on the median of the Hamming weights of codewords of any codebook  $\mathcal{C}$  that has high deniability. The intuition, as pictured in Figure 4, is that if many codewords have “high” Hamming weight, with non-negligible probability the Hamming weight of Willie’s observed vector  $\vec{y}_w$  will be above a carefully chosen threshold. This theorem (and Theorem 2 below) and our proof techniques are analogues (for the BSCs considered in this paper) of Theorem 2 in [2] (derived there for AWGN channels).

**Theorem 1.** *If more than half of the codewords in  $\mathcal{C}$  are of weight greater than  $c_1 \sqrt{n}$ , then the deniability of Alice’s codebook  $\mathcal{C}$  is less than  $1/2 + \frac{4p_w(1-p_w)}{c_1^2(1-2p_w)^2}$ .*

Theorem 2 below proves an upper bound on the throughput  $\tau$  of any code that simultaneously has high reliability and deniability. The proof technique follows standard information-theoretic converse arguments, though at one point we critically need to use the fact (proved in Theorem 1 above) that codes that have high deniability do not have too many codewords of high Hamming weight. Again, this theorem and its proof are analogues (for BSCs) of Theorem 2 in [2] (derived there for AWGN channels).

**Theorem 2.** *If a codebook  $\mathcal{C}$  is simultaneously  $(1-\epsilon)$ -deniable*

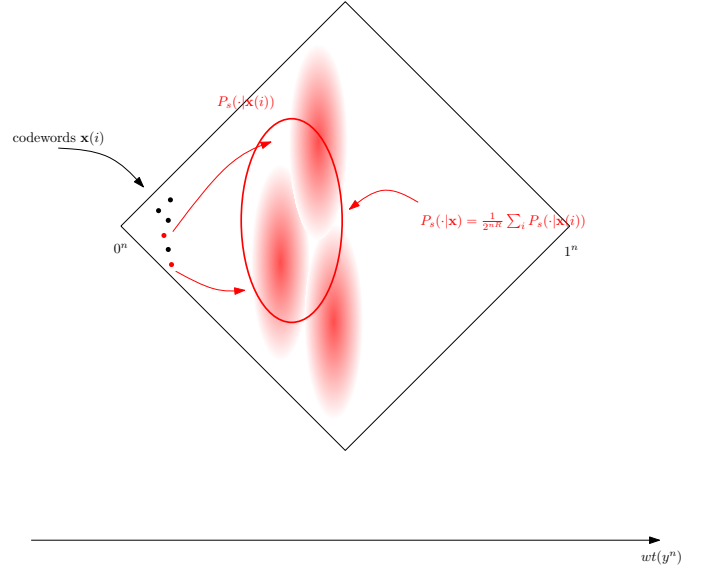


Fig. 3. The diamond at the top of this figure denotes the set (of size  $2^n$ ) of all possible  $\vec{y}_w$  that Willie may observe if Alice transmits a codeword uniformly at random from her codebook  $\mathcal{C}$  (her transmission status  $\mathbf{T} = 1$ ). Since codewords in our codebook  $\mathcal{C}$  have expected Hamming weight  $\rho n$ , the “typical”  $\vec{y}_w$  that Willie observes are of weight approximately  $(p_w * \rho)n$ . The curve at the bottom plots the probability distribution of observing  $\vec{y}_w$  of a particular weight. In this case (unlike in Figure 2) the probability distribution on  $\vec{y}_w$  is somewhat “lumpy”, since the probability that Willie observes a particular  $\vec{y}_w$  depends on the distribution of the Hamming distance between that particular  $\vec{y}_w$  and the set of codewords  $\vec{\mathbf{x}} \in \mathcal{C}$ . In particular, the probability distribution on  $\vec{y}_w$  equals  $\Pr_{\mathbf{M}, \vec{\mathbf{Z}}_b}(\vec{y}_w | \mathbf{T} = 1) = \frac{1}{2^n} \sum_{\vec{\mathbf{x}} \in \mathcal{C}} \Pr_{\vec{\mathbf{Z}}_b}(\vec{y}_w | \vec{\mathbf{x}})$ .

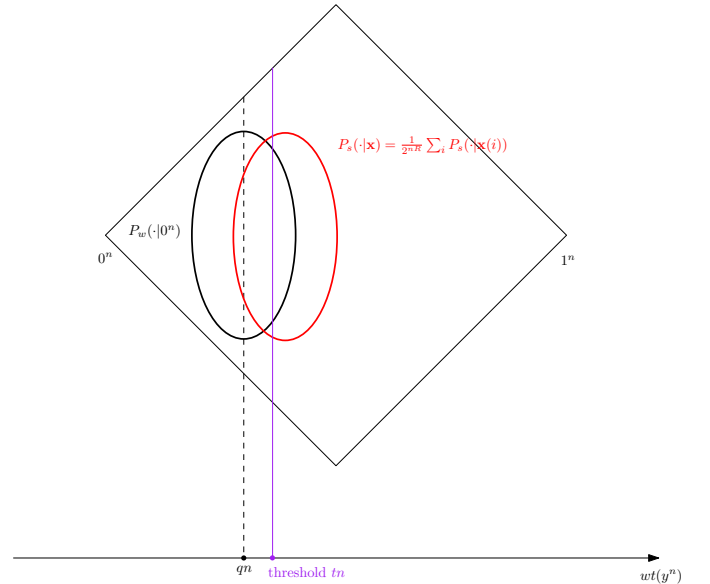


Fig. 4. If “too many” codewords in the codebook  $\mathcal{C}$  have “large” Hamming weight, the two probability distributions on  $\vec{y}_w$  corresponding to  $\mathbf{T} = 0$  and  $\mathbf{T} = 1$ , respectively  $\Pr_{\vec{\mathbf{Z}}_b}(\vec{y}_w | \mathbf{T} = 0)$  and  $\Pr_{\mathbf{M}, \vec{\mathbf{Z}}_b}(\vec{y}_w | \mathbf{T} = 1)$ , are “very different”. In this case, Willie can detect Alice’s transmission status fairly accurately just by outputting  $\hat{\mathbf{T}} = 1$  if the weight is above a carefully chosen threshold, and  $\hat{\mathbf{T}} = 0$  otherwise.

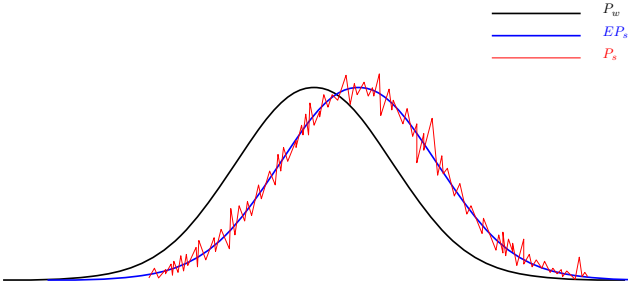


Fig. 5. Our proof that a random codebook  $\mathcal{C}$  chosen with the “right” parameters (number of codewords, expected weight of codewords) proceeds as follows. We need to demonstrate that the probability distributions  $\Pr_{\bar{\mathbf{z}}_b}(\vec{y}_w|\mathbf{T} = 0)$  and  $\Pr_{\mathcal{C}, \mathbf{M}, \bar{\mathbf{z}}_b}(\vec{y}_w|\mathbf{T} = 1)$  are “close” (in variational distance). However, since the latter distribution is complex (due to its dependence on the specific codebook  $\mathcal{C}$ ), we do this comparison in two stages. We first demonstrate that the probability distribution  $\Pr_{\bar{\mathbf{z}}_b}(\vec{y}_w|\mathbf{T} = 0)$  and the *ensemble distribution*  $\Pr_{\mathcal{C}, \mathbf{M}, \bar{\mathbf{z}}_b}(\vec{y}_w|\mathbf{T} = 1)$  (i.e. the weighted average over all possible codebooks  $\mathcal{C}$  of the latter distribution) are “close”. Next, we prove that with high probability over the choice of codebooks  $\mathcal{C}$ , the distribution of  $\Pr_{\mathcal{C}, \mathbf{M}, \bar{\mathbf{z}}_b}(\vec{y}_w|\mathbf{T} = 1)$  is tightly concentrated around its expectation  $\Pr_{\mathcal{C}, \mathbf{M}, \bar{\mathbf{z}}_b}(\vec{y}_w|\mathbf{T} = 1)$ .

and  $(1 - \epsilon)$ -reliable, the relative throughput is at most

$$\sqrt{p_w(1 - p_w)} \frac{1 - 2p_b}{1 - 2p_w} (1 - 2\epsilon)^{-3/2} \log \left[ \frac{1 - 2p_b}{p_b} \right]$$

Next we state and prove one of the main results of this work – namely, that randomly chosen codes (chosen from a suitable ensemble) are with high probability simultaneously highly reliable and highly deniable. This type of code is a significant improvement over the corresponding code constructions in [1, 2]. This is because the code constructions there require Alice and Bob to have access to common randomness (that is secret from Willie), whereas our codes are “public” (Willie knows exactly as much about the codebook as Bob does). In fact, the amount of common randomness required for the constructions in [1, 2] scales at least linearly in  $n$  – note that the throughput of reliable/deniable communication for the problem scales at most as  $\mathcal{O}(\sqrt{n})$ . This means that Alice burns through common randomness shared with Bob at a much faster pace than the throughput of the messages passed from her to Bob.

The code constructions in [1, 2] use this common randomness between Alice and Bob in the following way. First, they use the common randomness to coordinate which of an ensemble of possible codebooks Alice actually uses to communicate with Bob. Since Bob knows the common randomness, he knows which codebook Alice actually used. However, since the common randomness is kept secret from Willie, from his perspective, if Alice transmits a non-zero codeword, the probability distribution on his received  $\vec{y}_w$  is a *code ensemble average*  $\Pr_{\mathcal{C}, \mathbf{M}, \bar{\mathbf{z}}_b}(\vec{y}_w|\mathbf{T} = 1)$ . Using some elegant statistical properties of this ensemble average distribution, and the distribution  $\Pr_{\bar{\mathbf{z}}_b}(\vec{y}_w|\mathbf{T} = 0)$  (corresponding to Willie’s observations if Alice does not transmit anything), [1, 2] demonstrate that Willie is essentially unable

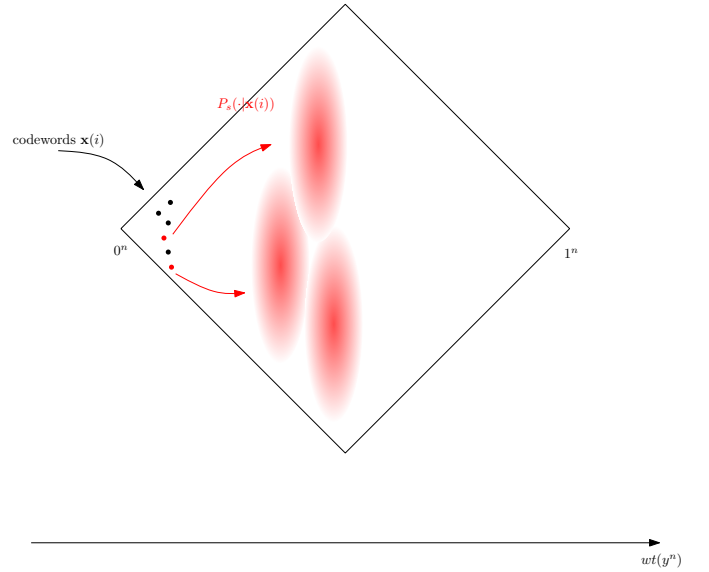


Fig. 6. Since the noise parameter  $p_b$  on the channel to Bob is smaller than the corresponding parameter  $p_w$  on the channel towards Willie, as long as the throughput is not too high, it can be shown that the sets of  $\vec{y}_w$  that are “typical” with respect to the corresponding transmitted codewords  $\vec{x}$  have relatively small intersection. Hence Bob can reliably decode by performing minimum distance decoding.

to learn anything about the binary random variable  $\mathbf{T}$ . Their proof can be essentially summarized in the statement that “the ensemble average codebook is highly deniable”. The challenge in extending their proof technique to a public codebook is that this proof says nothing about the existence of a single, public, highly deniable codebook.

Our key idea is to extend the analysis by proving that the *actual* distribution  $\Pr_{\mathbf{M}, \bar{\mathbf{z}}_b}(\vec{y}_w|\mathbf{T} = 1)$  on  $\vec{y}_w$  if Alice transmits a non-zero codeword is tightly concentrated about its ensemble average (this is visually depicted in Figure 5). However, our first “naïve” attempts in using standard concentration inequalities were unsuccessful, since for any  $\vec{y}_w$  the probability (averaged over Alice’s choice of message, channel noise, and over all codebooks) that Willie actually observes  $\vec{y}_w$  is exceedingly small (decaying at least exponentially in  $n$ ).

Hence we proceed indirectly. We first note that it suffices to prove that  $\Pr_{\mathbf{M}, \bar{\mathbf{z}}_b}(\vec{y}_w|\mathbf{T} = 1)$  converges point-wise to its ensemble average for “typical”  $\vec{y}_w$  (since the bulk of the probability mass of the ensemble average distribution falls in a certain range). For any  $\vec{y}_w$  in this range, we prove that expected number of codewords at a certain distance range (corresponding to the “typical” noise patterns  $\bar{\mathbf{z}}_w$ ) of each  $\vec{y}_w$  is super-polynomial. For random variables with such “large” (super-polynomial) expectations, standard arguments suffice to prove concentration with probability that is super-exponentially small in  $n$ . This allows us to show that with high probability over the ensemble average, a randomly chosen codebook satisfies the property that the number of codewords in “typical” Hamming shells around most “typical”  $\vec{y}_w$  are tightly concentrated around their expectations. Book-keeping

calculations then enable us to show that this concentration in the distance-distribution of codewords translate to a pointwise concentration (with super-exponential probability) of  $\Pr_{\mathbf{M}, \vec{\mathbf{Z}}_b}(\vec{\mathbf{Y}}_w | \mathbf{T} = 1)$  to its ensemble average. This technique allows us to bypass the problem of the small expected values<sup>6</sup> of the random variables of primary interest (the probability of observing specific channel outputs if a specific codebook is used), by focusing instead on random variables with “large expected values”<sup>7</sup> (numbers of codewords of certain “types”) that then enable us to recover the random variables of primary interest. One calculation that requires some care is that due to the low throughput of our codes (scaling as  $\mathcal{O}(\sqrt{n})$ ) we need to define our typical sets carefully, to simultaneously ensure that they are high probability sets, but are also not “too large”.

To complete the proof, we need to demonstrate that in fact a randomly chosen code (from the same ensemble as used to generate the highly deniable code above) is also highly reliable with sufficiently high probability (and hence a randomly chosen code is, with high probability, simultaneously high deniable and highly reliable). This follows from somewhat standard random coding arguments, if Bob decodes to the nearest codeword. Since the noise in the channel from Alice to Bob is smaller than the noise in the channel from Alice to Willie, the sets of  $\vec{\mathbf{y}}_b$  that are “typical” (with respect to channel noise  $\vec{\mathbf{Z}}_b$ ) with different transmitted codewords, have small intersection with each other. This is visually depicted in Figure 6. Hence Bob is able to decode even though Willie cannot even tell whether Alice is transmitting or not. One calculation that requires some care is due to the low expected weight of codewords (by construction chosen to be about  $\mathcal{O}(\sqrt{n})$ ), and hence the notion of “typicality decoding” has to be somewhat delicately defined.

We generate the ensemble of all codebooks  $\mathcal{C}$  containing  $2^{r\sqrt{n}}$  codewords of block-length  $n$ , with each codeword generated by choosing each bit to be 1 with probability Bernoulli( $\rho$ ), and sample codebooks from this distribution (we call such codebooks *random public codebooks*). In what follows, we set  $\rho$  to equal  $c_2/\sqrt{n}$ , where  $c_2$  is a code design parameter specified later.

**Theorem 3. SJ:** *For any  $p_w \in (, 1/2)$ , with probability greater than  $1 - 2^{-\theta(\sqrt{n})}$ , a random public codebook  $\mathcal{C}$  is simultaneously  $(1 - \epsilon)$ -reliable and  $(1 - \epsilon)$ -deniable.*

**Note 1:** One may notice that since we have not optimized all constants in our proof, we require not only that  $p_w > p_b$ , but in fact that  $p_w$  is rather close to  $1/2$  (though it still works for values bounded away from  $1/2$ ). It would be interesting to examine whether in fact any  $p_w > p_w$  would also work.

**Note 2:** It can be verified that if  $p_b$  is zero while  $p_b$  is not, then in fact a throughput that scales as  $\mathcal{O}(\sqrt{n} \log(n))$  is both reliably and deniably possible. (This does not contradict the outer bound in Theorem 2, since the higher order terms in the bound vanish, and the bound then becomes  $\mathcal{O}(\sqrt{n} \log(n))$ .)

<sup>6</sup>Or, as George Walker Bush put it, “the soft bigotry of low expectations”.

<sup>7</sup>Or, as Philip Pirrip might put it, we have “Great Expectations”.

In that case, code construction and proof of its properties proceeds as follows. Alice’s codebook comprises of *all* codewords of weight at most  $c_2\sqrt{n}$  (it can be verified that there are  $2^{\sqrt{n} \log(n)}$  such codewords). Since there is no noise on Bob’s channel, reliability is automatically guaranteed. As to deniability, it can be directly verified that the second term in the analogue of Equation (4) in the proof of Theorem 3 equals zero, considerably simplifying the proof.

## V. PROOF OF THEOREM 1:

Willie denotes the fraction of 1’s in his observed codeword  $\vec{\mathbf{Y}}_w$  by the random variable  $\mathbf{S}$ .

Note that if Alice does not transmit anything, then  $\vec{\mathbf{Y}}_w$  is purely the result of noise on the channel from Alice to Willie. In this case the expected value and variance of  $\mathbf{S}$  (over the randomness in the channel noise) are respectively

$$\mathbb{E}[\mathbf{S} | \mathbf{T} = 0] = p_w, \text{ and}$$

$$\text{Var}[\mathbf{S} | \mathbf{T} = 0] = \frac{1}{n} p_w (1 - p_w).$$

On the other hand, suppose Alice transmits a codeword  $\vec{\mathbf{X}}_m$  of weight  $\rho_0 n$ . In this case the expected value and variance of  $\mathbf{S}$  (over the randomness in the channel noise) are respectively

$$\mathbb{E}[\mathbf{S} | \vec{\mathbf{X}}_m] = p_w * \rho_0, \text{ and}$$

$$\text{Var}[\mathbf{S} | \vec{\mathbf{X}}_m] = \frac{1}{n} (p_w * \rho_0) (1 - p_w * \rho_0).$$

(Recall that  $p_w * \rho_0$  is defined as  $p_w(1 - \rho_0) + (1 - p_w)\rho_0$ .)

Willie chooses a *threshold*  $t$  as  $\frac{d}{\sqrt{n}}$ , where  $d = \sqrt{\frac{p_w(1-p_w)}{\alpha^*}}$ , and  $\alpha^*$  is a parameter to be defined later. He then sets his estimator  $Est_{\mathcal{C}}(\vec{\mathbf{Y}}_w)$  to equal 0 if  $\mathbf{S} < p_w + t$ , and 1 otherwise.

By Chebyshev’s inequality, we have

$$\begin{aligned} \alpha(Est_{\mathcal{C}}(\vec{\mathbf{Y}}_w)) &= \Pr_0(\mathbf{S} \geq p_w + t) \\ &\leq \Pr_0(|\mathbf{S} - p_w| \geq t) \\ &\leq \frac{p_w(1-p_w)}{nt^2} = \alpha^*. \end{aligned}$$

Similarly, for codewords of weight  $n\rho_0$  we have

$$\begin{aligned} \beta^{(m)}(Est_{\mathcal{C}}(\vec{\mathbf{Y}}_w)) &= \Pr_1^{(m)}(\mathbf{S} < p_w + t) \\ &\leq \Pr_1^{(m)}(|\mathbf{S} - p_w - \rho_0 + 2p_w\rho_0| \geq \rho_0 - 2p_w\rho_0 - t) \\ &= \Pr_1^{(m)}(|\mathbf{S} - p_w * \rho_0| \geq \rho_0 - 2p_w\rho_0 - t) \\ &\leq \frac{p_w + \rho_0 - 2p_w\rho_0 - (p_w + \rho_0 - 2p_w\rho_0)^2}{n(\rho_0 - 2p_w\rho_0 - t)^2}. \end{aligned} \tag{1}$$

For notational convenience, we let

$$\begin{aligned} A &= p_w(1 - p_w) \\ B &= p_w + \rho_0 - 2p_w\rho_0 - (p_w + \rho_0 - 2p_w\rho_0)^2 \\ C &= \rho_0(1 - 2p_w) \end{aligned}$$

So,

$$\alpha + \beta^{(m)} \leq \frac{1}{n} [At^{-2} + B(C - t)^{-2}]$$

Let  $f(t) = \frac{1}{n} [At^{-2} + B(C-t)^{-2}]$ , and let  $\frac{df(t)}{dt} = 0$ , we have

$$\begin{aligned} \frac{1}{n} [-2At^{-3} + 2B(C-t)^{-3}] &= 0 \\ -2At^{-3} + 2B(C-t)^{-3} &= 0 \\ 2B(C-t)^{-3} &= 2At^{-3} \\ \frac{C}{t} &= \left(\frac{B}{A}\right)^{\frac{1}{3}} + 1 \\ t &= \frac{C}{\left(\frac{B}{A}\right)^{\frac{1}{3}} + 1} \end{aligned}$$

Substitute  $\hat{t} = \frac{C}{\left(\frac{B}{A}\right)^{\frac{1}{3}} + 1}$  to  $f(t)$ ,

$$\begin{aligned} f(\hat{t}) &= \frac{1}{n} \left[ \frac{A \left(\left(\frac{B}{A}\right)^{\frac{1}{3}} + 1\right)^2}{C^2} + \frac{B}{C^2 \left(1 - \frac{1}{\left(\frac{B}{A}\right)^{\frac{1}{3}} + 1}\right)^2} \right] \\ &= \frac{1}{n} \left[ \frac{A \left(\left(\frac{B}{A}\right)^{\frac{1}{3}} + 1\right)^2}{C^2} + \frac{B \left(\left(\frac{B}{A}\right)^{\frac{1}{3}} + 1\right)^2}{C^2 \left(\frac{B}{A}\right)^{\frac{2}{3}}} \right] \\ &= \frac{1}{nC^2} \left[ \left(\frac{B}{A}\right)^{\frac{1}{3}} + 1 \right]^2 \left[ A + \frac{B}{\left(\frac{B}{A}\right)^{\frac{2}{3}}} \right] \\ &= \frac{A^{\frac{2}{3}}}{nC^2} \left[ \left(\frac{B}{A}\right)^{\frac{1}{3}} + 1 \right]^2 \left[ A^{\frac{1}{3}} + B^{\frac{1}{3}} \right] \end{aligned}$$

Note that for  $\rho_0 = c_1/\sqrt{n}$ , as  $n \rightarrow \infty$ ,  $nC^2 \rightarrow \infty$ ,  $\left(\frac{B}{A}\right)^{\frac{1}{3}} + 1 \rightarrow \text{const}$  and  $A^{\frac{1}{3}} + B^{\frac{1}{3}} \rightarrow \text{const}$ . This means,  $f(t) \rightarrow 0$  as  $n \rightarrow \infty$ . Thus,  $\rho_0 = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ .

More precisely, if we set  $\rho_0 = c_1 n^{-\delta}$ , where  $\delta \in (0, 1)$ . We have  $B \rightarrow A$  as  $n \rightarrow \infty$ , and

$$\lim_{n \rightarrow \infty} nC^2 = \begin{cases} \infty & \text{if } \delta \in (0, \frac{1}{2}) \\ c_1^2(1-2p_w)^2 & \text{if } \delta = \frac{1}{2} \\ 0 & \text{if } \delta \in (\frac{1}{2}, 1) \end{cases}.$$

So,

$$\lim_{n \rightarrow \infty} f(\hat{t}) = \begin{cases} 0 & \text{if } \delta \in (0, \frac{1}{2}) \\ \frac{8p_w(1-p_w)}{c_1^2(1-2p_w)^2} & \text{if } \delta = \frac{1}{2} \\ \infty & \text{if } \delta \in (\frac{1}{2}, 1) \end{cases}.$$

The deniability of Alice's codebook is equals to  $\alpha + \beta = \frac{1}{2^{rn}} \sum_{i=1}^{rn} (\alpha + \beta^{(m)})$ . We break this sum into two parts - terms involving codewords with weight less than or equal to  $c_1\sqrt{n}$  and those involving codewords of weight greater than  $c_1\sqrt{n}$ . The former terms are bounded by the previous analysis, while the deniability for the latter terms is at most 1. Further, since at most half off the codewords are of weight greater than  $c_1\sqrt{n}$ , we have

$$\alpha + \beta \leq 1/2 + \frac{4p_w(1-p_w)}{c_1^2(1-2p_w)^2}.$$

## VI. PROOF OF THEOREM 2

By Theorem 1, for a code to be  $(1-\epsilon)$ -deniable, at most  $1/2$  the codewords of  $\mathcal{C}$  are of weight at most  $\rho_\epsilon n$ , where

$$\rho_\epsilon = \sqrt{\frac{4p_w(1-p_w)}{(1/2-\epsilon)(1-2p_w)^2 n}}.$$

We denote this subset of codewords by  $\mathcal{C}'$ . We now bound the relative throughput of the code via parameters of this sub-code.

Bob's probability of decoding error when Alice is transmitting,  $P_{e,T=1}$ , (and thus also his overall probability of error  $P_e$ ) is at least  $1/2$  of  $P'_{e,T=1}$ . Here  $P'_{e,T=1}$  denotes Bob's probability of decoding error when Alice transmits a message using codebook  $\mathcal{C}'$  (instead of  $\mathcal{C}$ ). Recall that the relative throughput of a code was defined as the binary logarithm of the number of codewords, divided by  $\sqrt{n}$ . Since  $\mathcal{C}$  has at most twice the number of codewords that  $\mathcal{C}'$  does, if  $r$  and  $r'$  denote the relative throughputs of the code  $\mathcal{C}$  and the sub-code  $\mathcal{C}'$ , then  $r' \geq r - 1/\sqrt{n}$ . We use the apostrophe ' to denote random variables corresponding to the sub-code  $\mathcal{C}'$ . Then,

$$\begin{aligned} r' \sqrt{n} &= H(\mathbf{M}') \\ &= H(\mathbf{M}' | \hat{\mathbf{M}}') + I(\mathbf{M}'; \hat{\mathbf{M}}') \\ &\leq 1 + r' \sqrt{n} P'_{e,T=1} + I(\vec{\mathbf{X}}; \vec{\mathbf{Y}}) \\ &\leq 1 + r' \sqrt{n} P'_{e,T=1} + \sum_{i=1}^n (H(\mathbf{Y}_i) - H(\mathbf{Y}_i | \mathbf{X}_i)) \\ &\leq 1 + \sqrt{n} r' P'_{e,T=1} + n(H(p_b * \rho_\epsilon) - H(p_b)) \\ &= 1 + r' \sqrt{n} P'_{e,T=1} \\ &\quad + n \left[ \mathcal{D}(p_b \| p_b * \rho_\epsilon) + \rho_\epsilon (1 - 2p_b) \log \frac{1 - p_b * \rho_\epsilon}{p_b * \rho_\epsilon} \right]. \end{aligned}$$

Here, the first inequality holds due to Fano's inequality and the Data Processing inequality, the second inequality holds since the channel is memoryless, and the third inequality since  $\mathcal{C}'$  only has codewords of weight at most  $\rho_\epsilon n$  and hence  $H(p_b * \rho_\epsilon) \geq H(\mathbf{Y}_i)$ . Rearranging the terms of the above inequality and replacing  $P'_{e,T=1}$  by its upper bound of  $2\epsilon$ , we get

$$\begin{aligned} r' &\leq \frac{1 + n \left( \mathcal{D}(p_b \| p_b * \rho_\epsilon) + \rho_\epsilon (1 - 2p_b) \log \frac{1 - p_b * \rho_\epsilon}{p_b * \rho_\epsilon} \right)}{(1 - P'_{e,T=1}) \sqrt{n}} \\ &\leq \frac{1 + n \left( \mathcal{D}(p_b \| p_b * \rho_\epsilon) + \rho_\epsilon (1 - 2p_b) \log \frac{1 - p_b * \rho_\epsilon}{p_b * \rho_\epsilon} \right)}{(1 - 2\epsilon) \sqrt{n}} \end{aligned}$$

Now, by the Claims in the Appendix, we have

$$\mathcal{D}(p_b \| p_b * \rho_\epsilon) \leq \rho_\epsilon^2 \left( \frac{1}{p_b} + \frac{1}{1 - p_b} \right).$$

Further,  $\log \frac{1 - p_b * \rho_\epsilon}{p_b * \rho_\epsilon} \leq \log \frac{1 - p_b}{p_b}$ . Therefore, we have

$$r' \leq \frac{1 + n \left( \rho_\epsilon^2 \left( \frac{1}{p_b} + \frac{1}{1 - p_b} \right) + \rho_\epsilon (1 - 2p_b) \log \frac{1 - p_b}{p_b} \right)}{(1 - 2\epsilon) \sqrt{n}}.$$



Finally, we note that  $\rho_\epsilon = c_1/\sqrt{n}$ . Therefore, in the limit, as  $n$  grows without bound, we get

$$\begin{aligned} r &\leq \frac{c_1(1-2p_b)\log\frac{1-2p_b}{p_b}}{1-2\epsilon} \\ &= \sqrt{\frac{4p_w(1-p_w)(1-2p_b)^2\log^2\frac{1-2p_b}{p_b}}{(1/2-\epsilon)(1-2p_w)^2(1-2\epsilon)^2}} \\ &= \sqrt{p_w(1-p_w)}\frac{1-2p_b}{1-2p_w}(1-2\epsilon)^{-3/2}\log\left[\frac{1-2p_b}{p_b}\right]. \end{aligned}$$

## VII. PROOF OF THEOREM 3

In the two following subsections we argue respectively that (with overwhelming probability (super exponentially close to 1 as  $n$  increases without bound) over the randomness in the choice of the codebook  $\mathcal{C}$ ) a randomly chosen code is highly reliable, and also highly deniable. This then implies the existence of a single (publicly known) codebook that is simultaneously highly reliable and highly deniable.

### A. Reliability

Recall the codebook  $\mathcal{C}$  was generated by choosing  $2^r\sqrt{n}$  codewords, with each bit of each codeword generated *i.i.d.* according to Bernoulli( $\rho$ ). Bob uses a minimum-distance decoder.

If Alice did not transmit ( $\mathbf{T} = 0$ ) we define  $\mathbf{P}_e^{(0)}$  as the probability (over  $\tilde{\mathbf{Z}}_b$ ) of the event  $\{\hat{\mathbf{M}} \neq 0 | \mathbf{T} = 0\}$ . If Alice did transmit, then the error event corresponds to  $\{\hat{\mathbf{M}} \neq \mathbf{M} | \mathbf{T} = 1\}$ . This event includes two distinct scenarios – one corresponding to  $\hat{\mathbf{M}} \neq 0$  (Bob decoding to the wrong non-zero message), and the other corresponding to  $\hat{\mathbf{M}} = 0$  (Bob estimating that Alice did not transmit anything, even though Alice did transmit). We use  $\mathbf{P}_e^{(1)}$  to denote the probability (over  $\mathbf{M}, \tilde{\mathbf{Z}}_b$ ) of the first of these two scenarios  $\{\hat{\mathbf{M}} \neq 0, \hat{\mathbf{M}} \neq \mathbf{M} | \mathbf{T} = 1\}$ . We then use the observation that due to the properties of the zero codeword  $\tilde{\mathbf{0}}$ , and of minimum-distance decoders, the probability (over  $\mathbf{M}, \tilde{\mathbf{Z}}_b$ ) of Bob estimating  $\hat{\mathbf{M}} = 0$  even though  $\mathbf{T} = 1$  equals the probability (over  $\tilde{\mathbf{Z}}_b$ ) of Bob estimating  $\hat{\mathbf{M}} \neq 0$  even though  $\mathbf{T} = 0$ . Given this, we simplify our overall probability of error  $\mathbf{P}_e$  as the sum  $2\mathbf{P}_e^{(0)} + \mathbf{P}_e^{(1)}$  (the factor of 2 arises from our observation above).

The *error event*  $\mathcal{E}$  is defined as the union of the error events. By slight abuse of notation, we use  $\mathbf{P}_e(\mathcal{C})$  *etc* to denote the probabilities corresponding to specific codes  $\mathcal{C}$ .

Note that

$$\begin{aligned} \mathbf{P}_e^{(1)} &= \sum_{\mathcal{C}} \Pr(\mathcal{C}) \mathbf{P}_e^{(1)}(\mathcal{C}) \\ &= \sum_{\mathcal{C}} \Pr(\mathcal{C}) \sum_{m=1}^{2^{r\sqrt{n}}} \Pr(\mathbf{M} = m) \Pr(\mathcal{E} | \mathbf{M} = m, \mathcal{C} = \mathcal{C}) \\ &= \sum_{\mathcal{C}} \Pr(\mathcal{C}) \sum_{m=1}^{2^{r\sqrt{n}}} \frac{1}{2^{r\sqrt{n}}} \Pr(\mathcal{E} | \mathbf{M} = m, \mathcal{C} = \mathcal{C}) \\ &= \frac{1}{2^{r\sqrt{n}}} \sum_{m=1}^{2^{r\sqrt{n}}} \sum_{\mathcal{C}} \Pr(\mathcal{C}) \Pr(\mathcal{E} | \mathbf{M} = m, \mathcal{C} = \mathcal{C}) \\ &= \sum_{\mathcal{C}} \Pr(\mathcal{C}) \Pr(\mathcal{E} | \mathbf{M} = m, \mathcal{C} = \mathcal{C}) \\ &= \Pr_{\mathcal{C}}(\mathcal{E} | \mathbf{M} = m) \end{aligned}$$

So, without loss of generality, we assume the first message is sent. That is,  $\mathbf{P}_e^{(1)} = \Pr_{\mathcal{C}}(\mathcal{E} | \mathbf{M} = 1)$ . As is common in Shannon-theoretic achievability proofs, all subsequent calculations of probabilities in this subsection are also calculated over the randomness of the choice of specific codebook  $\mathcal{C}$ , besides the randomness in the channel noise  $\tilde{\mathbf{Z}}_b$ .

Bob's probability of decoding error can now be calculated in two steps. First, we focus on  $\mathbf{P}_e^{(1)}$ .

$$\mathbf{P}_e^{(1)} = \sum_{m:m \neq 1} \Pr[\mathbf{d}_H(\vec{\mathbf{x}}(1), \vec{\mathbf{y}}) \geq \mathbf{d}_H(\vec{\mathbf{x}}(m), \vec{\mathbf{y}})].$$

First, denote  $E_m^{(1)} = \{\mathbf{d}_H(\vec{\mathbf{x}}(1), \vec{\mathbf{y}}) \geq \mathbf{d}_H(\vec{\mathbf{x}}(m), \vec{\mathbf{y}})\}$ , where  $m \neq 1$  corresponds to the error event given that the message 1 is chosen. Then we have that

$$\begin{aligned} \mathbf{P}_e^{(1)} &= \sum_{\vec{\mathbf{x}} \in \{0,1\}^n} \Pr(\vec{\mathbf{X}}(1) = \vec{\mathbf{x}}) \left[ \Pr\left(\bigcup_{m=2}^{2^{r\sqrt{n}}} E_m^{(1)}\right)\right] \\ &\leq \sum_{\vec{\mathbf{x}} \in \{0,1\}^n} \Pr(\vec{\mathbf{X}}(1) = \vec{\mathbf{x}}) \left[ \sum_{m=2}^{2^{r\sqrt{n}}} \Pr(E_m^{(1)}) \right] \\ &= \sum_{m=2}^{2^{r\sqrt{n}}} \left[ \sum_{\vec{\mathbf{x}} \in \{0,1\}^n} \Pr(\vec{\mathbf{X}}(1) = \vec{\mathbf{x}}) \left( \Pr(E_m^{(1)}) \right) \right] \end{aligned}$$

We use  $\mathbf{T}_m^{(1)}$  to denote  $\text{supp}(\vec{\mathbf{x}}(1)) - \text{supp}(\vec{\mathbf{x}}(\mathbf{M}))$ . Here  $\text{supp}()$  denotes the *support* of a binary codeword, and  $-$  denotes the *symmetric difference* of the corresponding sets. This means  $\mathbf{T}_m^{(1)}$  is a random variable over the randomness in  $\mathbf{M}$ . For notational convenience, we use  $\mathbb{E}_{\vec{\mathbf{x}}(1)}(\cdot) =$

$\sum_{\vec{x} \in \{0,1\}^n} \Pr(\vec{\mathbf{X}}(1) = \vec{x}) (\cdot)$ . Then,

$$\begin{aligned} & \mathbb{E}_{\vec{x}(1)} \left[ \Pr(E_m^{(1)}) \right] \\ &= \mathbb{E}_{\vec{x}(1)} \left[ \Pr(\mathbf{d}_H(\vec{x}(1), \vec{y}) \geq \mathbf{d}_H(\vec{x}(m), \vec{y})) \right] \\ &= \mathbb{E}_{\vec{x}(1)} \left[ \Pr\left(\text{ratio of 1's of } \vec{\mathbf{Z}}_b \text{ in } \mathbf{T}_m^{(1)} \geq \frac{1}{2}\right) \right] \\ &= \mathbb{E}_{\mathbf{T}_m^{(1)}} \left[ \Pr\left(\text{ratio of 1's of } \vec{\mathbf{Z}}_b \text{ in } \mathbf{T}_m^{(1)} \geq \frac{1}{2}\right) \right] \\ &= \mathbb{E}_{\mathbf{T}_m^{(1)}} \left[ \sum_{j: j \geq \frac{1}{2} |\mathbf{T}_m^{(1)}|} \binom{|\mathbf{T}_m^{(1)}|}{j} p_b^j (1-p_b)^{|\mathbf{T}_m^{(1)}|-j} \right] \\ &\leq \mathbb{E}_{\mathbf{T}_m^{(1)}} \left[ e^{-2(\frac{1}{2}-p_b)^2 |\mathbf{T}_m^{(1)}|} \right]. \end{aligned}$$

with the last inequality following from the Chernoff bound.

Recall that we set  $\rho = \frac{c_2}{\sqrt{n}}$ . Also note that  $|\mathbf{T}_m^{(1)}| = |\text{supp}(\vec{x}(1))| + |\text{supp}(\vec{x}(m))| - 2|\text{supp}(\vec{x}(1)) \cap \text{supp}(\vec{x}(m))|$ . Then, we have  $\mu_m^{(1)}$ , defined as  $\mathbb{E}\left(|\mathbf{T}_m^{(1)}|\right)$ , equals  $c_2\sqrt{n} + c_2\sqrt{n} - 2c_2^2 = 2c_2(\sqrt{n} - c_2)$ . Also, since all  $\mu_m^{(1)}$  are equal, we use  $\mu^{(1)}$  to denote  $\mu_m^{(1)}$ . So,  $\Pr(|\mathbf{T}_m| < (1-\delta)\mu^{(1)}) \leq e^{-\frac{1}{2}\delta^2\mu^{(1)}}$ .

Hence, letting  $c_3 = 2(\frac{1}{2} - p_b)^2$ , we have

$$\begin{aligned} & \mathbb{E}_{\mathbf{T}_m^{(1)}} \left[ 2^{-c_3 |\mathbf{T}_m^{(1)}|} \right] \\ &= \sum_{|\mathbf{T}_m^{(1)}| < (1-\delta)\mu^{(1)}} \Pr\left(|\mathbf{T}_m^{(1)}|\right) e^{-c_3 t} + \sum_{|\mathbf{T}_m^{(1)}| \geq (1-\delta)\mu^{(1)}} \Pr\left(|\mathbf{T}_m^{(1)}|\right) e^{-c_3 t} \\ &\leq e^{-\frac{1}{2}\delta^2\mu^{(1)}} + e^{-c_3(1-\delta)\mu^{(1)}} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{P}_e^{(1)} &\leq \sum_{m=2}^{2^r\sqrt{n}} \left[ e^{-\frac{1}{2}\delta^2\mu^{(1)}} + e^{-c_3(1-\delta)\mu^{(1)}} \right] \\ &\leq 2^r\sqrt{n} \left[ e^{-\frac{1}{2}\delta^2\mu^{(1)}} + e^{-c_3(1-\delta)\mu^{(1)}} \right] \end{aligned}$$

Noting that  $\mu_m^{(1)}$  equals  $2c_2(\sqrt{n} - c_2)$ , we note that for any  $r < \frac{\mu^{(1)} \log e}{\sqrt{n}} \min\{\frac{1}{2}\delta^2, c_3(1-\delta)\}$ , (the right hand side of which converges to  $2c_2(\log e)(\min\{\frac{1}{2}\delta^2, c_3(1-\delta)\})$ ), implies that  $\mathbf{P}_e^{(1)}$  goes to zero superpolynomially quickly in  $n$  (more specifically, as  $2^{-\theta(\sqrt{n})}$ ).

Similarly, the probability of error given  $\mathbf{T} = 0$  is

$$\begin{aligned} \mathbf{P}_e^{(0)} &= \Pr[\exists i, \mathbf{d}_H(\vec{0}, \vec{y}) \geq \mathbf{d}_H(\vec{x}(m), \vec{y})] \\ &= \Pr[\exists i, \mathbf{wt}_H(\vec{z}) \geq \mathbf{d}_H(\vec{x}(m), \vec{z})] \end{aligned}$$

Denote  $E_m^{(0)} = \{\mathbf{wt}_H(\vec{z}) \geq \mathbf{d}_H(\vec{x}(m), \vec{z})\}$  to be the error event when Alice does not communicate with Bob. We also denote  $\mathbf{T}_m^{(0)} = \text{supp}(\vec{x}(m))$ . Then, we have

$$\begin{aligned} \mathbf{P}_e^{(0)} &= \Pr\left(\bigcup_i E_m^{(0)}\right) \\ &\leq \sum_{m=1}^{2^r\sqrt{n}} \sum_{\vec{x} \in \{0,1\}^n} \Pr(\vec{\mathbf{X}}(m) = \vec{x}) \Pr(E_m^{(0)}) \\ &= 2^r\sqrt{n} \mathbb{E}_{\vec{x}(1)} \left[ \Pr(E_1^{(0)}) \right]. \end{aligned}$$

Note that since  $\rho = \frac{c_2}{\sqrt{n}}$ ,  $\mu_m^{(0)}$ , defined as  $\mathbb{E}\left(|\mathbf{T}_m^{(0)}|\right)$ , equals  $c_2\sqrt{n}$ . We use  $\mu^{(0)}$  to denote  $\mu_m^{(0)}$ . Hence

$$\begin{aligned} & \mathbb{E}_{\vec{x}(1)} \left[ \Pr(E_1^{(0)}) \right] \\ &= \mathbb{E}_{\vec{x}(1)} \left( \text{ratio of 1's of } \vec{\mathbf{Z}}_b \text{ in } \mathbf{T}_1^{(0)} \geq \frac{1}{2} \right) \\ &= \mathbb{E}_{\mathbf{T}_m^{(0)}} \left( \text{ratio of 1's in } \vec{\mathbf{Z}}_b \text{ in } \mathbf{T}_1^{(0)} \geq \frac{1}{2} \right) \\ &= \mathbb{E}_{\mathbf{T}_m^{(0)}} \left[ \sum_{j: j \geq \frac{1}{2} |\mathbf{T}_1^{(0)}|} \binom{|\mathbf{T}_1^{(0)}|}{j} p_b^j (1-p_b)^{|\mathbf{T}_1^{(0)}|-j} \right] \\ &\leq \mathbb{E}_{\mathbf{T}_m^{(0)}} \left[ e^{-2(1-p_b)^2 |\mathbf{T}_m^{(0)}|} \right]. \end{aligned}$$

Hence,

$$\begin{aligned} & \mathbb{E}_{\mathbf{T}_m^{(0)}} \left[ e^{-c_3 |\mathbf{T}_m^{(0)}|} \right] \\ &\leq e^{-\frac{1}{2}\delta^2\mu^{(0)}} + e^{-c_3(1-\delta)\mu^{(0)}}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{P}_e^{(0)} &\leq \sum_{m=1}^{2^r\sqrt{n}} \left[ e^{-\frac{1}{2}\delta^2\mu^{(0)}} + e^{-c_3(1-\delta)\mu^{(0)}} \right] \\ &\leq 2^r\sqrt{n} \left[ e^{-\frac{1}{2}\delta^2\mu^{(0)}} + e^{-c_3(1-\delta)\mu^{(0)}} \right] \end{aligned}$$

Noting that  $\mu_m^{(0)}$  equals  $c_2\sqrt{n}$ , we note that for any  $r < \frac{\mu^{(0)} \log e}{\sqrt{n}} \min\{\frac{1}{2}\delta^2, c_3(1-\delta)\}$ , (the right hand side of which converges to  $c_2(\log e)(\min\{\frac{1}{2}\delta^2, c_3(1-\delta)\})$ ), implies that  $\mathbf{P}_e^{(0)}$  goes to zero superpolynomially quickly in  $n$  (more specifically, as  $2^{-\theta(\sqrt{n})}$ ).

Now recall that the overall probability of error equals  $2\mathbf{P}_e^{(0)} + \mathbf{P}_e^{(1)}$ . Hence, for random codes, Bob's overall probability of error decays to zero as  $2^{-\theta(\sqrt{n})}$ . But this probability *also* includes the randomness over choice of codebook  $\mathcal{C}$ . Hence by Markov's inequality this means that at least a  $1 - 2^{-\theta(\sqrt{n})}$  fraction of all possible codes are  $(1 - \epsilon)$ -deniable.

## B. Deniability

We now prove that a random code  $\mathcal{C}$  *also* has overwhelming probability of being highly deniable. This requires substantially more work than the proof of reliability in the previous subsection, which broadly followed somewhat standard achievability techniques.

For notational convenience, for each  $\vec{y} \in \{0, 1\}^n$  let  $\Pr_w^n(\vec{y})$  denote the probability (over the random variable  $\vec{Z}_w$ ) that Willie observes  $\vec{y}$  given that Alice's transmission status is 0 (she transmits nothing). Analogously, let  $\Pr_s^n(\vec{y})$  denote the probability (over the random variables  $\mathbf{M}, \vec{Z}_w$ ) that Willie observes  $\vec{y}$  given that Alice's transmission status is 1 (to reiterate – this probability is calculated as an average over both the random variables  $\mathbf{M}$  and  $\vec{Z}_w$ , but is for a *specific* code  $\mathcal{C}$ , and hence is not necessarily “smooth”). Finally, let  $\mathbb{E}(\Pr_s^n(\vec{y}))$  denote the “smoothed” version of  $\Pr_s^n(\vec{y})$ , specifically, averaged over all codebooks generated according to the probability distribution specified in the achievability scheme.

Recall that a code  $\mathcal{C}$  is  $(1-\epsilon)$ -deniable if for every estimator  $Est_{\mathcal{C}}(\cdot)$  of Willie,

$$\alpha(Est_{\mathcal{C}}(\cdot)) + \beta(Est_{\mathcal{C}}(\cdot)) \geq 1 - \epsilon. \quad (2)$$

But by standard statistical arguments (reprinted in [2] as Fact 1), (2) is implied by the condition that

$$\mathbb{V}(\Pr_w^n, \Pr_s^n) \leq \epsilon. \quad (3)$$

Here use  $\mathbb{V}(\Pr_1, \Pr_2)$  to denote the *variational distance* between any two probability distributions  $\Pr_1(\vec{y})$  and  $\Pr_2(\vec{y})$ , i.e.,  $\mathbb{V}(\Pr_1, \Pr_2)$  is defined as

$$\frac{1}{2} \left( \sum_{\vec{y} \in \{0,1\}^n} |\Pr_1(\vec{y}) - \Pr_2(\vec{y})| \right).$$

But by the triangle inequality,

$$\mathbb{V}(\Pr_w^n, \Pr_s^n) \leq \mathbb{V}(\Pr_w^n, \mathbb{E}_{\mathcal{C}}(\Pr_s^n)) + \mathbb{V}(\mathbb{E}_{\mathcal{C}}(\Pr_s^n), \Pr_s^n). \quad (4)$$

Also, we note that  $\Pr_w^n(\vec{y})$  corresponds to the  $n$ -letter distribution induced by  $n$  Bernoulli- $(p_w)$  random variables. Similarly, the “smoothed” distribution  $\mathbb{E}_{\mathcal{C}}(\Pr_s^n)$  corresponds to the  $n$ -letter distribution induced by  $n$  Bernoulli- $(p_w * \rho)$  random variables.

By further standard statistical arguments (reprinted in [2] as Facts 2 and 3), we have

$$\begin{aligned} \mathbb{V}(\Pr_w^n, \Pr_s^n) &\leq \sqrt{\frac{1}{2} \mathcal{D}(\Pr_w^n \| \Pr_s^n)} \\ &= \sqrt{\frac{n}{2} \mathcal{D}(\Pr_w \| \Pr_s)} \end{aligned}$$

Hence, using the Taylor series bound on the Kullback-Leibler divergence shown in Claim 2, and recalling that  $\rho$  is set to equal  $c_2/\sqrt{n}$ , we have that

$$\mathbb{V}(\Pr_w^n, \Pr_s^n) \leq c(p_w) \rho \sqrt{\frac{n}{2}} \leq c(p_w) \frac{c_2}{\sqrt{2}}, \quad (5)$$

where  $c(p_w) = \sqrt{\frac{(1-2p_w)^2}{p_w} - \frac{(1-2p_w)^2}{1-p_w}}$ .

Up to this point, the proof is similar to the proof in [2]. The remainder of this paper focuses on the challenging task of bounding the second term in (4). Clearly this second term need not be small for specific codes  $\mathcal{C}$  – a “bad” codebook will not

behave like its expectation.<sup>8</sup> Slightly more precisely, we aim to show that with high probability over  $\mathcal{C}$ ,  $\mathbb{V}(\mathbb{E}_{\mathcal{C}}(\Pr_s^n), \Pr_s^n)$  is “small”.

We now proceed as follows.

We define the *typical type classes* of  $\vec{Y}$  as  $\mathcal{T}_{\{w_y\}}(\vec{Y}) \triangleq \{\vec{y} : \mathbf{wt}_H(\vec{y}) = w_y n \in (n(p_w * \rho - \Delta_Y), n(p_w * \rho + \Delta_Y))\}$ . Then,

$$\begin{aligned} &\mathbb{V}(\mathbb{E}_{\mathcal{C}}(\Pr_s^n), \Pr_s^n) \\ &= \frac{1}{2} \sum_{\vec{y}} |\Pr_s(\vec{y}) - \mathbb{E}_{\mathcal{C}} \Pr_s(\vec{y})| \\ &= \frac{1}{2} \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{Y})} |\Pr_s(\vec{y}) - \mathbb{E}_{\mathcal{C}} \Pr_s(\vec{y})| \\ &\quad + \frac{1}{2} \sum_{\vec{y} \notin \mathcal{T}_{\{w_y\}}(\vec{Y})} |\Pr_s(\vec{y}) - \mathbb{E}_{\mathcal{C}} \Pr_s(\vec{y})| \end{aligned}$$

Next, we define  $d_{ij} = \frac{|\{k: x_k=i, y_k=j\}|}{n}$  for  $i, j \in \{0, 1\}$ . We also (for each typical  $\vec{y}$ ) define the *typical type classes* of  $\vec{X}$  with respect to  $\vec{y}$  as  $\mathcal{T}_{\{d, w_x\}}(\vec{X}|\vec{y}) \triangleq \{\vec{x} : \mathbf{d}_H(\vec{x}, \vec{y}) = dn \in (n(p_w - \Delta_{Y|X}), n(p_w + \Delta_{Y|X})), \mathbf{wt}_H(\vec{x}) = w_x n \in (n\rho(1 - \Delta_X), n\rho(1 + \Delta_X))\}$ , and  $\mathcal{T}_{(d, w_x)}(\vec{X}|\vec{y}) \triangleq \{\vec{x} : \mathbf{d}_H(\vec{x}, \vec{y}) = dn, \mathbf{wt}_H(\vec{x}) = w_x n\}$ . Then the  $\Pr_s(\vec{y})$  can be written in terms of channel parameters, and sizes of type classes, as follows:

$$\begin{aligned} \Pr_s(\vec{y}) &= \sum_{\vec{x} \in \mathcal{C}} p(\vec{y}|\vec{x})p(\vec{x}) \\ &= \sum_{\vec{x} \in \mathcal{T}_{\{d, w_x\}}(\vec{X}|\vec{y}) \cap \mathcal{C}} p(\vec{y}|\vec{x})p(\vec{x}) \\ &\quad + \sum_{\vec{x} \notin \mathcal{T}_{\{d, w_x\}}(\vec{X}|\vec{y}) \cap \mathcal{C}} p(\vec{y}|\vec{x})p(\vec{x}) \quad (6) \end{aligned}$$

Also, the first term in RHS of equation (6)

$$\begin{aligned} &\sum_{\vec{x} \in \mathcal{T}_{\{d, w_x\}}(\vec{X}|\vec{y}) \cap \mathcal{C}} p(\vec{y}|\vec{x})p(\vec{x}) \\ &= \sum_{dn, w_x n} \sum_{\vec{x} \in \mathcal{T}_{(d, w_x)}(\vec{X}|\vec{y}) \cap \mathcal{C}} p(\vec{y}|\vec{x})p(\vec{x}) \\ &= \frac{1}{2^r \sqrt{n}} \sum_{dn, w_x n} \left| \mathcal{T}_{(d, w_x)}(\vec{X}|\vec{y}) \cap \mathcal{C} \right| p(\vec{y}|\vec{x}) \end{aligned}$$

To be able to use concentration inequalities to prove that the actual number of elements in each “typical” type class we consider is close to its expectation, we would first like to bound from below the expected size of each typical type class, and show that this size is super-linear (which would allow for very tight concentration). We note that  $\mathbb{E}_{\mathcal{C}} \left( \left| \mathcal{T}_{(d, w_x)}(\vec{X}|\vec{y}) \cap \mathcal{C} \right| \right) =$

<sup>8</sup>For instance, if the codebook has only very high weight codewords, Theorem 1 already indicates that such a code cannot be highly deniable. The point is, of course, that the *probability* (over the ensemble of random codebooks) of such a code instantiating is small.

$\Pr_{\mathcal{C}} \left( \vec{\mathbf{X}} \in \mathcal{T}_{(d,w_x)}(\vec{\mathbf{X}}|\vec{\mathbf{y}}) \right) |C|$ , for a given  $\vec{\mathbf{y}} \in \mathcal{T}_{\{d,w_x\}}(\vec{\mathbf{X}}|\vec{\mathbf{y}})$ ,

$$\begin{aligned}
& \Pr_{\mathcal{C}}(\vec{\mathbf{X}} \in \mathcal{T}_{(d,w_x)}(\vec{\mathbf{X}}|\vec{\mathbf{y}})) \\
&= \binom{(d_{01} + d_{11})n}{d_{11}n} r^{d_{11}n} (1-r)^{d_{01}n} \\
&\quad \binom{(d_{10} + d_{00})n}{d_{10}n} r^{d_{10}n} (1-r)^{d_{00}n} \\
&\geq \frac{1}{(n+1)^2} 2^{(d_{01}+d_{11})nH\left(\frac{d_{11}}{d_{01}+d_{11}}\right)} \\
&\quad 2^{(d_{10}+d_{00})nH\left(\frac{d_{10}}{d_{10}+d_{00}}\right)} \\
&\quad r^{(d_{10}+d_{11})n} (1-r)^{(d_{00}+d_{01})n} \\
&= \frac{1}{(n+1)^2} 2^{nH(X_{\mathcal{T}}|Y_{\mathcal{T}})} 2^{-n(\mathcal{D}(X_{\mathcal{T}}\|X)+H(X_{\mathcal{T}}))} \\
&= \frac{1}{(n+1)^2} 2^{-n(I(X_{\mathcal{T}};Y_{\mathcal{T}})+\mathcal{D}(X_{\mathcal{T}}\|X))}
\end{aligned}$$

Hence we would first like to bound from below  $\Pr_{\mathcal{C}} \left( \vec{\mathbf{X}} \in \mathcal{T}_{(d,w_x)}(\vec{\mathbf{X}}|\vec{\mathbf{y}}) \right)$ , which is equivalent to bounding from above  $I(X_{\mathcal{T}}; Y_{\mathcal{T}}) + \mathcal{D}(X_{\mathcal{T}}\|X)$  over the ‘‘typical’’ types.

sions, we note that

$$\begin{aligned}
& \max_{T_{X|Y}, T_Y} I(X_{\mathcal{T}}; Y_{\mathcal{T}}) \\
&= \max_{T_{\vec{\mathbf{X}}}, T_{\vec{\mathbf{Y}}}} H(Y_{\mathcal{T}}) - H(Y_{\mathcal{T}}|X_{\mathcal{T}}) \\
&= H(p_w * \rho + \Delta_Y) - H(p_w - \Delta_{Y|X}) \\
&= (p_w * \rho + \Delta_Y) \log \frac{1}{p_w * \rho + \Delta_Y} \\
&\quad + (1p_w * \rho - \Delta_Y) \log \frac{1}{1 - p_w * \rho - \Delta_Y} \\
&\quad (p_w - \Delta_{Y|X}) \log \frac{1}{p_w - \Delta_{Y|X}} \\
&\quad - (1 - p_w + \Delta_{Y|X}) \log \frac{1}{1 - p_w + \Delta_{Y|X}} \\
&= \Delta_Y \log \frac{1}{p_w * \rho + \Delta_Y} - \Delta_Y \log \frac{1}{1 - p_w * \rho - \Delta_Y} \\
&\quad + \Delta_{Y|X} \log \frac{1}{p_w - \Delta_{Y|X}} - \Delta_{Y|X} \log \frac{1}{1 - p_w + \Delta_{Y|X}} \\
&\quad + (p_w * \rho) \log \frac{1}{p_w * \rho + \Delta_Y} \\
&\quad + (1 - p_w * \rho) \log \frac{1}{1 - p_w * \rho + \Delta_Y} \\
&\quad - p_w \log \frac{1}{p_w - \Delta_{Y|X}} \\
&\quad - (1 - p_w) \log \frac{1}{1 - p_w + \Delta_{Y|X}} \\
&= \Delta \log \frac{(1 - p_w * \rho - \Delta)(1 - p_w + \Delta)}{(p_w * \rho + \Delta)(p_w - \Delta)} \\
&\quad + \mathcal{D}(p_w * \rho \| p_w * \rho + \Delta) + H(p_w * \rho) \\
&\quad - \mathcal{D}(p_w \| p_w - \Delta) - H(p_w) \\
&= \Delta \log \frac{(1 - p_w * \rho - \Delta)(1 - p_w + \Delta)}{(p_w * \rho + \Delta)(p_w - \Delta)} \\
&\quad + \frac{\Delta^2}{2} \left( \frac{1}{p_w * \rho} + \frac{1}{1 - p_w * \rho} \right) \\
&\quad - \frac{\Delta^2}{2} \left( \frac{1}{p_w} + \frac{1}{1 - p_w} \right) \\
&\quad + \mathcal{O}(\Delta^3) \\
&\quad + \mathcal{D}(p_w \| p_w * \rho) + \rho(1 - 2p_w) \log \frac{1 - p_w * \rho}{p_w * \rho} \\
&= \Delta \log \frac{(1 - p_w * \rho - \Delta)(1 - p_w + \Delta)}{(p_w * \rho + \Delta)(p_w - \Delta)} \\
&\quad + \rho(1 - 2p_w) \log \frac{1 - p_w * \rho}{p_w * \rho} \\
&\quad + \mathcal{O}(\Delta^2) + \mathcal{O}(\rho^2),
\end{aligned}$$

Using the bounds obtained in the Appendix on KL-divergences and entropy functions via Taylor series expan-

Also,

$$\begin{aligned}
& \max_{T_X|Y} \mathcal{D}(X_{\mathcal{T}}\|X) \\
&= \mathcal{D}(\rho(1 + \Delta_X)\|\rho) \\
&= -H(\rho(1 + \Delta_X)) + H(\rho) + \rho\Delta_X \log \frac{1 - \rho}{\rho} \\
&= -\rho^2\Delta_X^2 \left( \frac{1}{\rho} + \frac{1}{1 - \rho} \right) \\
&\quad -\rho\Delta_X \log \frac{1 - \rho(1 + \Delta_X)}{\rho(1 + \Delta_X)} + \rho\Delta_X \log \frac{1 - \rho}{\rho} \\
&= -\rho^2\Delta_X^2 \left( \frac{1}{\rho} + \frac{1}{1 - \rho} \right) \\
&\quad +\rho\Delta_X \log \frac{(1 - \rho)(1 + \Delta_X)}{1 - \rho(1 + \Delta_X)} \\
&= -\rho^2\Delta_X^2 \left( \frac{1}{\rho} + \frac{1}{1 - \rho} \right) \\
&\quad +\rho\Delta_X \log \frac{1 - \rho + \Delta_X - \rho\Delta_X}{1 - \rho - \rho\Delta_X} \\
&\leq \rho\Delta_X \log \left( 1 + \frac{\Delta_X}{1 - \rho(1 + \Delta_X)} \right) \\
&\leq \rho\Delta_X \frac{\Delta_X}{1 - \rho(1 + \Delta_X)} \\
&= \frac{\rho\Delta_X^2}{1 - \rho(1 + \Delta_X)} \\
&= \mathcal{O}(\rho\Delta_X^2).
\end{aligned}$$

Now, note that by Chernoff's bound,

$$\begin{aligned}
& \Pr \left( \left| \frac{\mathbf{wt}_H(\vec{y})}{n} - p_w * \rho \right| > p_w * \rho + \Delta_Y \right) \\
&< 2e^{-\frac{1}{2}\Delta_Y^2 n},
\end{aligned}$$

and

$$\begin{aligned}
& \Pr \left( \left| \frac{\mathbf{d}_H(\vec{x}, \vec{y})}{n} - p_w \right| > p_w + \Delta_{Y|X} \right) \\
&< 2e^{-\frac{1}{2}\Delta_{Y|X}^2 n}.
\end{aligned}$$

Letting  $\Delta = \Delta_Y = \Delta_{Y|X} = \sqrt{\frac{2 \ln \frac{2}{\epsilon_2}}{n}} = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ , we have

$$\Pr \left( \left| \frac{\mathbf{wt}_H(\vec{y})}{n} - p_w * \rho \right| > p_w * \rho + \Delta_Y \right) < \epsilon_2,$$

and

$$\Pr \left( \left| \frac{\mathbf{d}_H(\vec{x}, \vec{y})}{n} - p_w \right| > p_w + \Delta_{Y|X} \right) < \epsilon_2.$$

Similarly, letting  $\Delta_X = \sqrt{\frac{3 \ln \frac{2}{\epsilon_3}}{n\rho}} = \sqrt{\frac{3 \ln \frac{2}{\epsilon_3}}{c_2\sqrt{n}}}$ , we have

$$\begin{aligned}
& \Pr(|\mathbf{wt}_H(\vec{x}) - n\rho| > n\Delta_X\rho) \\
&< 2e^{-\frac{1}{3}\Delta_X^2 n\rho} \\
&= \epsilon_3
\end{aligned}$$

So, defining  $c_4$  as  $\sqrt{2 \ln \frac{2}{\epsilon_2}}$ , we have

$$\begin{aligned}
& \max I(X_{\mathcal{T}}; Y_{\mathcal{T}}) + \mathcal{D}(X_{\mathcal{T}}\|X) \\
&\leq \frac{c_4}{\sqrt{n}} \log \frac{(1 - p_w * \rho - \frac{c_4}{\sqrt{n}})(1 - p_w + \frac{c_4}{\sqrt{n}})}{(p_w * \rho + \frac{c_4}{\sqrt{n}})(p_w - \frac{c_4}{\sqrt{n}})} \\
&\quad + \frac{c_2}{\sqrt{n}} (1 - 2p_w) \log \frac{1 - p_w * \rho}{p_w * \rho} \\
&\quad + \mathcal{O}\left(\frac{1}{n}\right)
\end{aligned}$$

Denoting by  $c_5$  the constant  $\sqrt{\frac{3 \ln \frac{2}{\epsilon_3}}{c_2}}$ , we have  $\max \mathcal{D}(x_{\mathcal{T}}\|X) = \mathcal{O}\left(\frac{1}{n}\right)$ .

Let  $c_6 = c_4 \log \frac{(1 - p_w * \rho - \frac{c_4}{\sqrt{n}})(1 - p_w + \frac{c_4}{\sqrt{n}})}{(p_w * \rho + \frac{c_4}{\sqrt{n}})(p_w - \frac{c_4}{\sqrt{n}})} + c_2(1 - 2p_w) \log \frac{1 - p_w * \rho}{p_w * \rho}$ , so  $\mathbb{E}_{\mathcal{C}}(|\mathcal{T}_{(d, w_x)}(\vec{\mathbf{X}}|\vec{y}) \cap C|) \geq \frac{1}{(n+1)^2} 2^{-c_6\sqrt{n} + \mathcal{O}(1)} 2^{r\sqrt{n}}$ . Therefore, for  $r > c_6$ ,  $\mathbb{E}_{\mathcal{C}}(|\mathcal{T}_{(d, w_x)}(\vec{\mathbf{X}}|\vec{y}) \cap C|) \geq 2^{c_7\sqrt{n}}$  for some  $c_7 > 0$ . Hence, we have

$$\begin{aligned}
& \Pr \left[ \left| |\mathcal{T}_{(d, w_x)}(\vec{\mathbf{X}}|\vec{y}) \cap C| - \mathbb{E}_{\mathcal{C}}(|\mathcal{T}_{(d, w_x)}(\vec{\mathbf{X}}|\vec{y}) \cap C|) \right| > \right. \\
&\quad \left. \epsilon_4 \mathbb{E}_{\mathcal{C}}(|\mathcal{T}_{(d, w_x)}(\vec{\mathbf{X}}|\vec{y}) \cap C|) \right] \\
&< 2 \exp \left( -\frac{1}{2} \epsilon_4^2 2^{c_7\sqrt{n}} \right)
\end{aligned}$$

This means,  $|\mathcal{T}_{(d, w_x)}(\vec{\mathbf{X}}|\vec{y}) \cap C|$  is highly concentrated around  $\mathbb{E}_{\mathcal{C}}(|\mathcal{T}_{(d, w_x)}(\vec{\mathbf{X}}|\vec{y}) \cap C|)$  for each "typical"  $\vec{y}$ . Note that,

$$\begin{aligned}
& \mathbb{E}_{\mathcal{C}}(\Pr_s(\vec{y})) \\
&= \sum_{\mathcal{C}} \Pr(C) \sum_{\vec{x} \in \mathcal{T}_{\{d, w_x\}}(\vec{\mathbf{X}}|\vec{y}) \cap C} p(\vec{y}|\vec{x})p(\vec{x}) \\
&= \sum_{\mathcal{C}} \Pr(C) \sum_{dn, w_x n} \sum_{\vec{x} \in \mathcal{T}_{(d, w_x)}(\vec{\mathbf{X}}|\vec{y}) \cap C} p(\vec{y}|\vec{x})p(\vec{x}) \\
&= \sum_{\mathcal{C}} \Pr(C) \sum_{dn, w_x n} |\mathcal{T}_{(d, w_x)}(\vec{\mathbf{X}}|\vec{y}) \cap C| p(\vec{y}|\vec{x})p(\vec{x}) \\
&= \sum_{dn, w_x n} \sum_{\mathcal{C}} \Pr(C) |\mathcal{T}_{(d, w_x)}(\vec{\mathbf{X}}|\vec{y}) \cap C| p(\vec{y}|\vec{x})p(\vec{x}) \\
&= \sum_{dn, w_x n} \mathbb{E}(|\mathcal{T}_{(d, w_x)}(\vec{\mathbf{X}}|\vec{y}) \cap C|) p(\vec{y}|\vec{x})p(\vec{x})
\end{aligned}$$

Therefore, for all  $\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{Y})$  and  $\vec{x} \in \mathcal{T}_{\{dn, w_x n\}}(\vec{\mathbf{X}}|\vec{y})$ ,

with probability super-exponentially close to 1,

$$\begin{aligned}
& \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{Y})} \left| \sum_{\vec{x} \in \mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \right. \\
& \quad \left. - \sum_C \Pr(C) \sum_{\vec{x} \in \mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \right| \\
&= \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{Y})} \left| \sum_{dn, w_x n} \sum_{\vec{x} \in \mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \right. \\
& \quad \left. - \sum_C \Pr(C) \sum_{dn, w_x n} \sum_{\vec{x} \in \mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \right| \\
&= \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{Y})} \left| \sum_{dn, w_x n} |\mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{y})| p(\vec{y}|\vec{x})p(\vec{x}) \right. \\
& \quad \left. - \sum_{dn, w_x n} \mathbb{E}(|\mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{y})|) p(\vec{y}|\vec{x})p(\vec{x}) \right| \\
&\leq \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{Y})} \sum_{dn, w_x n} \\
& \quad \left| |\mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{y})| - \mathbb{E}(|\mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{y})|) \right| \\
& \quad p(\vec{y}|\vec{x})p(\vec{x}) \\
&\leq \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{Y})} \sum_{dn, w_x n} \\
& \quad \epsilon_4 \mathbb{E}(|\mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{y})|) p(\vec{y}|\vec{x})p(\vec{x}) \\
&= \epsilon_4 \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{Y})} \sum_C \Pr(C) \\
& \quad \sum_{\vec{x} \in \mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \\
&< \epsilon_4 \tag{7}
\end{aligned}$$

It remains for us to show that for  $\vec{y} \notin \mathcal{T}_{\{w_y\}}(\vec{Y})$  or  $\vec{x} \notin \mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{y})$ , the contribution to the variational distance is small. Note that,

$$\begin{aligned}
& \frac{1}{2} \sum_{\vec{y}} |\Pr_s(\vec{y}) - \mathbb{E}_C \Pr_s(\vec{y})| \\
&= \frac{1}{2} \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{Y})} \left| \sum_{\vec{x} \in \mathcal{T}_{\{d, w_x\}}(\vec{X}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \right. \\
& \quad \left. \sum_C \Pr(C) \sum_{\vec{x} \in \mathcal{T}_{\{d, w_x\}}(\vec{X}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \right| \\
& \quad + \frac{1}{2} \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{Y})} \left| \sum_{\vec{x} \notin \mathcal{T}_{\{d, w_x\}}(\vec{X}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \right. \\
& \quad \left. \sum_C \Pr(C) \sum_{\vec{x} \notin \mathcal{T}_{\{d, w_x\}}(\vec{X}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \right| \\
& \quad + \frac{1}{2} \sum_{\vec{y} \notin \mathcal{T}_{\{w_y\}}(\vec{Y})} |\Pr_s(\vec{y}) - \mathbb{E}_C \Pr_s(\vec{y})|
\end{aligned}$$

Note that,

$$\begin{aligned}
& \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{Y})} \left| \sum_{\vec{x} \notin \mathcal{T}_{\{d, w_x\}}(\vec{X}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \right. \\
& \quad \left. \sum_C \Pr(C) \sum_{\vec{x} \notin \mathcal{T}_{\{d, w_x\}}(\vec{X}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \right| \\
& \quad + \sum_{\vec{y} \notin \mathcal{T}_{\{w_y\}}(\vec{Y})} |\Pr_s(\vec{y}) - \mathbb{E}_C \Pr_s(\vec{y})| \\
&\leq \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{Y})} \sum_{\vec{x} \notin \mathcal{T}_{\{d, w_x\}}(\vec{X}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \\
& \quad + \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{Y})} \sum_C \Pr(C) \\
& \quad \sum_{\vec{x} \notin \mathcal{T}_{\{d, w_x\}}(\vec{X}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \\
& \quad + \sum_{\vec{y} \notin \mathcal{T}_{\{w_y\}}(\vec{Y})} \Pr_s(\vec{y}) + \sum_{\vec{y} \notin \mathcal{T}_{\{w_y\}}(\vec{Y})} \mathbb{E}_C \Pr_s(\vec{y})| \\
&\leq \Pr(\vec{Y} \notin \mathcal{T}_{\{w_y\}}(\vec{Y}) \text{ or } \vec{X} \notin \mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{Y})) \\
& \quad + \mathbb{E}_C \left[ \Pr(\vec{Y} \notin \mathcal{T}_{\{w_y\}}(\vec{Y}) \text{ or } \vec{X} \notin \mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{Y})) \right]
\end{aligned}$$

Also,

$$\begin{aligned}
& \mathbb{E}_C \left[ \Pr(\vec{Y} \notin \mathcal{T}_{\{w_y\}}(\vec{Y}) \text{ or } \vec{X} \notin \mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{Y})) \right] \\
&\leq \mathbb{E}_C \left[ \Pr(\vec{Y} \notin \mathcal{T}_{\{w_y\}}(\vec{Y})) \right] \\
& \quad + \mathbb{E}_C \left[ \Pr(\vec{X} \notin \mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{Y})|\vec{Y}) \right] \\
&= \mathbb{E}_C \left[ \Pr(\vec{Y} \notin \mathcal{T}_{\{w_y\}}(\vec{Y})) \right] \\
& \quad + \Pr(\vec{X} \notin \mathcal{T}_{\{dn, w_x n\}}(\vec{X}|\vec{Y})|\vec{Y}) \\
&\leq \epsilon_2 + \epsilon_2 + \epsilon_3 \\
&= 2\epsilon_2 + \epsilon_3 \tag{8}
\end{aligned}$$

And by inequality (7) and (8), we have

$$\begin{aligned}
& 2\epsilon_2 + \epsilon_3 + \epsilon_4 \\
& > \mathbb{E}_C \left[ \Pr(\vec{\mathbf{Y}} \notin \mathcal{T}_{\{w_y\}}(\vec{\mathbf{Y}}) \text{ or } \vec{\mathbf{X}} \notin \mathcal{T}_{\{dn, w_x n\}}(\vec{\mathbf{X}}|\vec{\mathbf{Y}})) \right] \\
& \quad + \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{\mathbf{Y}})} \left| \sum_{\vec{x} \in \mathcal{T}_{\{dn, w_x n\}}(\vec{\mathbf{X}}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \right. \\
& \quad \left. - \sum_C \Pr(C) \sum_{\vec{x} \in \mathcal{T}_{\{dn, w_x n\}}(\vec{\mathbf{X}}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \right) \Big| \\
& \geq \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{\mathbf{Y}})} \left| \mathbb{E}_C \Pr_s(\vec{y}) - \sum_{\vec{x} \in \mathcal{T}_{\{dn, w_x n\}}(\vec{\mathbf{X}}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \right| \\
& \geq \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{\mathbf{Y}})} \mathbb{E}_C \Pr_s(\vec{y}) \\
& \quad - \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{\mathbf{Y}})} \sum_{\vec{x} \in \mathcal{T}_{\{dn, w_x n\}}(\vec{\mathbf{X}}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \\
& > 1 - \epsilon_2 - \sum_{\vec{y} \in \mathcal{T}_{\{w_y\}}(\vec{\mathbf{Y}})} \sum_{\vec{x} \in \mathcal{T}_{\{dn, w_x n\}}(\vec{\mathbf{X}}|\vec{y})} p(\vec{y}|\vec{x})p(\vec{x}) \\
& > 1 - \epsilon_2 \\
& \quad - \Pr(\vec{\mathbf{Y}} \in \mathcal{T}_{\{w_y\}}(\vec{\mathbf{Y}}) \text{ and } \vec{\mathbf{X}} \in \mathcal{T}_{\{dn, w_x n\}}(\vec{\mathbf{X}}|\vec{\mathbf{Y}})) \\
& = 1 - \epsilon_2 \\
& \quad - \left[ 1 - \Pr(\vec{\mathbf{Y}} \notin \mathcal{T}_{\{w_y\}}(\vec{\mathbf{Y}}) \text{ or } \vec{\mathbf{X}} \notin \mathcal{T}_{\{dn, w_x n\}}(\vec{\mathbf{X}}|\vec{\mathbf{Y}})) \right] \\
& = \Pr(\vec{\mathbf{Y}} \notin \mathcal{T}_{\{w_y\}}(\vec{\mathbf{Y}}) \text{ or } \vec{\mathbf{X}} \notin \mathcal{T}_{\{dn, w_x n\}}(\vec{\mathbf{X}}|\vec{\mathbf{Y}})) - \epsilon_2
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \Pr(\vec{\mathbf{Y}} \notin \mathcal{T}_{\{w_y\}}(\vec{\mathbf{Y}}) \text{ or } \vec{\mathbf{X}} \notin \mathcal{T}_{\{dn, w_x n\}}(\vec{\mathbf{X}}|\vec{\mathbf{Y}})) \\
& < 3\epsilon_2 + \epsilon_3 + \epsilon_4 \tag{9}
\end{aligned}$$

Hence by combining (7), (8) and (9), with probability super-exponentially close to 1, the variational distance  $\mathbb{V}(\mathbb{E}_C(\Pr_s^n), \Pr_s^n)$  satisfies

$$\begin{aligned}
& \frac{1}{2} \sum_{\vec{y}} |\Pr_s(\vec{y}) - \mathbb{E}_C \Pr_s(\vec{y})| \\
& < \frac{1}{2} [\epsilon_4 + 2\epsilon_2 + \epsilon_3 + 3\epsilon_2 + \epsilon_3 + \epsilon_4] \\
& = \frac{1}{2} [5\epsilon_2 + 2\epsilon_3 + 2\epsilon_4]. \tag{10}
\end{aligned}$$

Finally, substituting (10) and (5) into (4) and using (2) gives us that the deniability is at least

$$\alpha(\text{Est}_C(\cdot)) + \beta(\text{Est}_C(\cdot)) \geq 1 - \frac{1}{2} [5\epsilon_2 + 2\epsilon_3 + 2\epsilon_4] + c(p_w) \frac{\epsilon_2}{\sqrt{2}}. \tag{11}$$

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## APPENDIX

**Claim 1.** The Taylor Series expansion of  $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \mathcal{O}(x^4)$

*Proof.* By examining the Taylor Series expansion of the logarithmic function. ■

**Claim 2.**

$$\mathcal{D}(p||p+x) \leq x^2 \left( \frac{1}{p} + \frac{1}{1-p} \right) + \mathcal{O}(x^3)$$

*Proof.*

$$\begin{aligned} \mathcal{D}(p||p+x) &= p \log \frac{p}{p+x} + (1-p) \log \frac{1-p}{1-p-x} \\ &= -p \log \left( 1 + \frac{x}{p} \right) - (1-p) \log \left( 1 - \frac{x}{1-p} \right) \\ &= -x \left( \frac{x}{p} - \frac{x^2}{2p^2} + \theta_1 \right) \\ &\quad - (1-x) \left( -\frac{x}{1-p} - \frac{x^2}{2(1-p)^2} + \theta_2 \right) \\ &= \frac{x^2}{2} \left( \frac{1}{p} + \frac{1}{1-p} \right) \\ &\quad + \frac{x^3}{3} \left( \frac{1}{(1-p)^2} - \frac{1}{p^2} \right) + \mathcal{O}(x^4) \end{aligned}$$

**Corollary 1.**

$$\mathcal{D}(p||p*x) \leq x^2(1-2p)^2 \left( \frac{1}{p} + \frac{1}{1-p} \right) + \mathcal{O}(x^3)$$

*Proof.* Note that  $p*x = p + x(1-2p)$ , and Claim 2. ■

**Claim 3.**

$$H(p+x) - H(p) = \mathcal{D}(p||p+x) + x \log \frac{1-p-x}{p+x}$$

*Proof.*

$$\begin{aligned} H(p+x) - H(p) &= -(p+x) \log(p+x) - (1-p-x) \log(1-p-x) \\ &\quad + p \log p + (1-p) \log(1-p) \\ &= p \log \frac{p}{p+x} + (1-p) \log \frac{1-p}{1-p-x} \\ &\quad + x \log \frac{1-p-x}{p+x} \\ &= \mathcal{D}(p||p+x) + x \log \frac{1-p-x}{p+x} \end{aligned}$$

**Corollary 2.**

$$H(p*x) - H(p) = \mathcal{D}(p||p*x) + x(1-2p) \log \frac{1-p*x}{p*x}$$

**Claim 4.**

$$\mathcal{D}(p+x||p) = -H(p+x) + H(p) + x \log \frac{1-p}{p}$$

*Proof.*

$$\begin{aligned} \mathcal{D}(p+x||p) &= (p+x) \log \frac{p+x}{p} + (1-p-x) \log \frac{1-p-x}{1-p} \\ &= -H(p+x) \\ &\quad + (p+x) \log \frac{1}{p} + (1-p-x) \log \frac{1}{1-x} \\ &= -H(p+x) + H(p) + x \log \frac{1-p}{p}. \end{aligned}$$

■