On erasure correction coding for streaming

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***Abstract*—We consider packet erasure correction coding for a streaming system where specific information needs to be decoded by specific deadlines, in order to ensure uninterrupted playback at the receiver. In our previous work [1], we gave a capacity-achieving code construction for the case of a fixed number of erasures. In this work, we consider a sliding window erasure pattern where the number of erasures within windows of size above some threshold is upper bounded by a fraction of the window size, modeling a constraint on burstiness of the channel. We lower bound the rates achievable by our previous code construction as a fraction of the capacity region, which approaches to one as the window size threshold and the initial playout delay increase simultaneously.**

I. Introduction

We consider the problem of coding for streaming data over a packet erasure channel. For uninterrupted playback at the receiver, specific packets need to be decoded by specific deadlines. The code is designed to work under a set of possible erasure patterns, the realization of which is unknown a priori to the encoder.

In our previous work [1], by modeling this problem as a network erasure correction problem, we characterized the capacity region under z erasures with *a priori* unknown locations. We also presented an intra-session coding scheme that achieves the capacity region. The streaming problem is modeled as an erasure correction problem on a network where the receivers have a nested structure, i.e. the set of packets received by each receiver contains the set of packets received by its predecessor. Each link in the network represents a unit time, and each receiver in the network corresponds to a deadline and demands the packets which are to be decoded by that deadline.

In this paper, we consider a sliding window erasure model, which is characterized by two parameters, erasure rate p and a window size threshold T. We consider erasure patterns where the number of erasures in any window of size at least T is upper bounded by a fraction p of the window size. Such erasure patterns have a long term erasure rate bounded by p, and do not contain erasure bursts of length greater than pT.

We consider the intra-session coding scheme of [1] under the sliding window erasure model. We lower bound the ratio of the rate region of this coding scheme to the capacity region by a function of p, T, and the initial playout delay m_1. We establish that this function approaches to one for $m_1 \gg T \gg 1$. Other than the code rate, the coding scheme is independent of the parameters of the erasure model, and as such convenient to adapt.

In other related work, Martinian *et al*. [2], [3] provide constructions of streaming codes that minimize the delay required to correct erasure bursts of given length.

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II. Model and problem description

A. Sliding Window Erasure Model

This erasure model characterizes a class of possible erased subsets. It models systems in which erasures occur with a long-term rate p and excessively long erasure bursts are rare enough that we do not code for them. For instance, if each erasure occurs independently with probability p, the probability of a long erasure burst decreases exponentially with length. This motivates the following sliding window erasure model, which upper bounds the number of successive erasures:

Definition 1: An erasure pattern is called a sliding window erasure pattern for a given fraction p and threshold $T \in \mathbb{Z}^+$ if, for every $t \geq T$, no more than tp out of any t consecutive packets can be erased.

Note that under a sliding window erasure pattern with parameters p and T, erasures occur with a long term rate less than p, and the length of the erasure bursts are cotrolled by the threshold T. The following definition classifies possible unerased sets under the sliding window erasure model:

Definition 2: Let $A = \{a_1, a_2, \ldots, a_n\}$ be an ordered set. A subset B of A is called (p, T)-unerased subset of A, if the inequality

$$|B \cap \{a_{i+1}, a_{i+2}, \ldots, a_{i+t}\}| \geq (1-p)t \qquad (1)$$

is satisfied for all positive integer pairs (t, i) satisfying $t \geq T$, and $0 \leq i \leq n - t$.

B. Network erasure correction problem

Consider a streaming system where at each time step one packet of unit size is transmitted, and the receiver needs to decode specific independent messages $\{M_1, M_2, \ldots, M_n\}$ at given time steps $\{m_1, m_2, \ldots, m_n\}$ respectively. As described in [1], this can be viewed as an erasure correction problem on a *3-layer nested* network with one source and n sinks $\{t_1, t_2, \ldots, t_n\}$, constructed as follows and illustrated in Figure 1:

- $\mathcal{I} = \{l_1, l_2, ..., l_{m_n}\}$ is the set of middle layer links.
- The source is connected to all the links in $\mathcal{I}$.
- Sink t_i is connected to links $l_1, \ldots, l_{m_i}$.

- All links have unit capacity.
- Only the links in $\mathcal{I}$ can be erased.

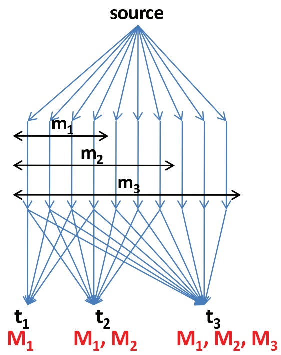

Fig. 1. 3-layer nested-network topology with three sinks.

III. Coding scheme

A. Intra-Session Coding

An intra-session coding scheme is one in which no coding occurs across information demanded by different receivers. For a given intra-session coding scheme, let q_i^j denote the amount of information corresponding to message M_j transmitted on the link l_i. A rate vector $(u_1, u_2, \ldots, u_n)$ is achievable under an erasure pattern by this intrasession coding scheme if the inequalities

$$\forall j : 1 \le j \le n \qquad u_j \le \sum_{i \in P \cap \{1,\ldots,m_i\}} q_i^j, \tag{2}$$

$$\forall i : 1 \le i \le m_n \qquad \sum_{j=1}^{n} q_i^j \le 1, \tag{3}$$

are satisfied for every permissible unerased set $P \subseteq \mathcal{I}$ under this erasure pattern. We assume that the packet size is large enough to accomodate an appropriate generic or random linear erasure code.

B. "As Uniform As Possible" Intra-Session Coding Scheme

In [1] we define the following intra-session coding scheme which assigns the rate for each successive receiver as uniformly as possible subject to capacity constraints imposed by assignments for previous receivers. This coding scheme resembles the water-filling process, as we allocate the packets of each receiver to the links in the upstream of the receiver equally as long as the links are not saturated by the previous assignments. For a given rate vector $(u_1, u_2, ..., u_n)$, we define a corresponding lower triangular $n \times n$ rate allocation matrix T, along with auxiliary variables $t_{i,j} \triangleq \sum_{k=j}^{i} T_{k,j}$, $d_{i,j} \triangleq \sum_{k=1}^{j}(m_k - m_{k-1})T_{i,k}$, and s_i, by Algorithm 1: Note that $T_{i,s_i} < T_{i,s_i+1} = T_{i,s_i+2} = \ldots = T_{i,i}$.

Definition 3: A rate vector $(u_1, u_2, ..., u_n)$ is called allocable if Algorithm 1 does not return any error and the corresponding allocation matrix T is non-negative.

Algorithm 1

$T_{1,1} = \frac{u_1}{m_1}, t_{1,1} = \frac{u_1}{m_1}, d_{1,1} = u_1, s_1 = 0, m_0 = 0$
for $i = 2 \to n$ **do**
  {allocation for sink i on links $l_{m_{j-1}+1}, \ldots, l_{m_j}$}
  $d_{i,0} = 0$
  $j = 1$
  while $1 - t_{i-1,j} < \frac{u_i - d_{i,j-1}}{m_i - m_{j-1}}$ **do**
    $T_{i,j} = 1 - t_{i-1,j}$
    $t_{i,j} = \sum_{k=j}^{i} T_{k,j}$
    $d_{i,j} = \sum_{k=1}^{j}(m_k - m_{k-1})T_{i,k}$
    $j \leftarrow j + 1$
    if $j > i$ or $u_i \le d_{i,j}$ **then**
      return error {rate vector is unallocable}
    end if
  end while
  $s_i = j - 1$ {the uniform portion follows}
  while $j \le i$ **do**
    $T_{i,j} = \frac{u_i - d_{i,s_i}}{m_i - m_{s_i}}$
    $t_{i,j} = \sum_{k=j}^{i} T_{k,j}$
    $d_{i,j} = \sum_{k=1}^{j}(m_k - m_{k-1})T_{i,k}$
    $j \leftarrow j + 1$
  end while
end for

Definition 4: Given an allocable rate vector $(u_1, u_2, ..., u_n)$, the "as uniform as possible" intra-session coding scheme is defined by the allocation

$$q_i^j = T_{j,k} \quad \forall i : m_{k-1} < i \le m_k. \tag{4}$$

Note that this coding scheme depends only on the rate vector $\vec{u}$ and the set of deadlines.

We observe that under the "as uniform as possible" intrasession coding scheme the amount of information transmitted on the middle layer of the network is monotone:

Lemma 1: If $(u_1, u_2, ..., u_n)$ is allocable, then the corresponding allocation matrix T satisfies:

$$T_{i,j} \le T_{i,j+1} \quad \forall i, j : 1 \le i \le n, 1 \le j < i$$

Proof: See the appendix. ■

IV. Main Result

We will verify the efficiency of the "as uniform as possible" intra-session coding scheme defined in Section III-B. The following lemma states that under the sliding window erasure, the amount of the information loss is controlled by the parameter p for certain information allocations:

Lemma 2: Let $A = \{a_1, a_2, \ldots, a_n\}$ be an ordered set of nonnegative real numbers with $a_1 \ge a_2 \ge \ldots \ge a_n$ and $a_1 = a_2 = \ldots = a_T = a$. Let $||X||$ denote the sum of elements in an arbitrary finite set of real numbers X. Let B be a (p, T)-unerased subset of A under some sliding window erasure with parameters p and T. Then,

$$||B|| \ge (1 - p)||A||.$$

Proof: See the appendix. ∎

Note that the monotonicity of the numbers a_i is crucial as Lemma 1 states that the amount of the information allocated for a certain message M_i on the middle layer of the network is monotone under the "as uniform as possible" intrasession coding scheme. The following lemma establishes that the equally allocated part of a message has a length of at least T under the "as uniform as possible" intrasession coding for a constant multiple of a vector inside the capacity region of the network under the erasure-free case. Hence, by Lemma 2, the rate of the erased information for a particular message M_i will not be greater than p.

Lemma 3: Let $V = \{(v_1, v_2, \dots, v_n) | \sum_{i=1}^{k} v_i \leq m_k \quad \forall k : 1 \leq k \leq n\}$. Let $\vec{v} \in V$. The rate vector $\frac{1}{1+\log(\frac{m_1+T}{m_1})}\vec{v}$ is achievable by the "as uniform as possible" intra-session coding, in a such way that the corresponding allocation q satisfies:

$$q^i_{m_i-T+1} = q^i_{m_i-T+2} = \dots = q^i_{m_i} \quad \forall i : 1 \leq i \leq n. \tag{5}$$

Proof: See the appendix. ∎

Lemma 4 and Lemma 5 compares the capacity regions of the "as uniform as possible" intrasession coding scheme and any other coding scheme to an intermediate region V, which is the capacity region under the erasure-free case:

Lemma 4: Let U be the erasure correction capacity region under some sliding window erasure with parameters p and T. Let $V = \{(v_1, v_2, \dots, v_n) | \sum_{i=1}^{k} v_i \leq m_k \quad \forall k : 1 \leq k \leq n\}$. Then

$$U \subset (1 - p + \frac{1}{T})V. \tag{6}$$

Proof: See the appendix. ∎

Lemma 5: Let W be the erasure correction capacity region obtained by "as uniform as possible" intrasession coding under sliding window erasure with parameters p and T. Let $V = \{(v_1, v_2, \dots, v_n) | \sum_{i=1}^{k} v_i \leq m_k \quad \forall k : 1 \leq k \leq n\}$. Then

$$(1-p)\frac{1}{1+\log(\frac{m_1+T}{m_1})}V \subset W. \tag{7}$$

Proof: See the appendix. ∎

The following theorem states that the erasure correction capacity region of the "as uniform as possible" intra-session coding contains a constant multiple of that of any other coding scheme:

Theorem 1: Let U be the erasure correction capacity region under sliding window erasure with parameters p and T. Let W be the erasure correction capacity region obtained by the "as uniform as possible" intrasession coding under sliding window erasure with parameters p and T. Then,

$$\frac{1-p}{(1+\log(\frac{m_1+T}{m_1}))(1-p+\frac{1}{T})}U \subset W. \tag{8}$$

Proof: Applying Lemma 4 and Lemma 5, we obtain (8). ∎

Let $\lambda = \sup\{x \in \mathbb{R} : xU \subset W\}$. As λ is the ratio of the two regions U and W, it measures how close the "as uniform as possible" intrasession coding scheme to the optimal coding scheme. Theorem 1 implies that

$$\lambda \geq \frac{1-p}{(1+\log(\frac{m_1+T}{m_1}))(1-p+\frac{1}{T})}.$$

Note that for $T \gg 1$, and $m_1 \gg T$ we have

$$\frac{1-p}{(1+\log(\frac{m_1+T}{m_1}))(1-p+\frac{1}{T})} \approx 1,$$

which implies that $\lambda \approx 1$ for large values of T and large values of m_1 compared to T.

Appendix

Proof of Lemma 1: Let's prove the following statements simultaneously by induction:

$$\left.\begin{aligned} T_{i,j} \leq T_{i,j+1} \quad \forall i,j : 1 \leq i \leq n, 1 \leq j < i, \\ t_{i,j} \geq t_{i,j+1} \quad \forall i,j : 1 \leq i \leq n, 1 \leq j < i. \end{aligned}\right\} \tag{9}$$

The statements hold trivially for $(i, j) = (1, 1)$. Let the statements hold for all (i, j) before (k, l) in lexicographical order. Let's now verify in three cases that (9) is satisfied for $(i, j) = (k, l)$:

Case(1): $l < s_k$:

By construction, we have:

$$\begin{aligned} T_{k,l} = 1 - t_{k-1,l}, T_{k,l+1} = 1 - t_{k-1,l+1}, \\ t_{k,l} = t_{k,l+1} = 1. \end{aligned}$$

Clearly, $t_{k,l} \leq t_{k,l+1}$. By induction hypothesis, we have $t_{k-1,l} \geq t_{k-1,l+1}$. Hence

$$T_{k,l} = 1 - t_{k-1,l} \leq 1 - t_{k-1,l+1} = T_{k,l+1}.$$

Case(2): $l = s_k$:

By construction, we have:

$$T_{k,l} = 1 - t_{k-1,l} < \frac{u_k - d_{k,l-1}}{m_k - m_{l-1}}, \tag{10}$$

$$T_{k,l+1} = \frac{u_k - d_{k,l}}{m_k - m_l} \leq 1 - t_{k-1,l+1}. \tag{11}$$

Hence,

$$t_{k,l} = t_{k-1,l} + T_{k,l} = 1 \geq t_{k-1,l+1} + T_{k,l+1} = t_{k,l+1}.$$

From (11) we get:

$$\begin{aligned} T_{k,l+1} &= \frac{u_k - d_{k,l}}{m_k - m_l} \\ &= \frac{u_k - d_{k,l-1} - (m_l - m_{l-1})T_{k,l}}{m_k - m_l}. \end{aligned} \tag{12}$$

Using (10) we obtain:

$$T_{k,l}(m_k - m_{l-1}) < u_k - d_{k,l-1}. \tag{13}$$

Combining (12) and (13) we get:

$$T_{k,l+1} > T_{k,l}.$$

Case(3): $l > s_k$:
By construction, we have:

$$T_{k,l} = T_{k,l+1},$$
$$t_{k,l} = T_{k,l} + t_{k-1,l}, t_{k,l+1} = T_{k,l+1} + t_{k-1,l+1}.$$

By induction hypothesis, $t_{k-1,l} \geq t_{k-1,l+1}$. Hence,

$$t_{k,l} \geq t_{k,l+1}.$$

■

Proof of Lemma 2: Let $B = \{a_{k_1}, a_{k_2}, \ldots, a_{k_m}\}$, where $k_1 < k_2 < \ldots < k_m$. Let $q = 1-p$. Let s be the largest integer satisfiying $k_s \leq T$. Let's prove by induction that

$$a_{k_1} + a_{k_2} + \ldots + a_{k_r} \geq q(a_1 + a_2 + \ldots + a_{\lfloor \frac{r}{q} \rfloor}) + (r - q\lfloor \frac{r}{q} \rfloor) a_{\lfloor \frac{r}{q} \rfloor + 1} \quad (14)$$

is satisfied for any r with $m \geq r \geq s$.
For $r = s$, we have

$$\begin{aligned}\sum_{i=1}^{s} a_{k_i} &= sa \\ &= aq\lfloor \frac{s}{q} \rfloor + (s - q\lfloor \frac{s}{q} \rfloor)a \\ &\geq q(a_1 + a_2 + \ldots + a_{\lfloor \frac{s}{q} \rfloor}) + (s - q\lfloor \frac{s}{q} \rfloor) a_{\lfloor \frac{s}{q} \rfloor + 1}.\end{aligned}$$

Let (14) be satisfied for some $r \geq s$. As $k_{r+1} > T$, and B satisfies (1) we have

$$\begin{aligned}r = |B \cap \{a_1, a_2, \ldots, a_{k_{r+1}-1}\}| &\geq (1-p)(k_{r+1} - 1) \\ &= q(k_{r+1} - 1),\end{aligned}$$

which is equivalent to

$$k_{r+1} \leq 1 + \frac{r}{q}.$$

As k_{r+1} is an integer we have:

$$k_{r+1} \leq 1 + \lfloor \frac{r}{q} \rfloor,$$

which implies

$$a_{k_{r+1}} \geq a_{\lfloor \frac{r}{q} \rfloor + 1}. \quad (15)$$

As a_i is monotone, using (15) and the induction hypothesis we get

$$\begin{aligned}\sum_{i=1}^{r+1} a_{k_i} \geq& q \sum_{i=1}^{\lfloor \frac{r}{q} \rfloor} a_i + (r - q\lfloor \frac{r}{q} \rfloor) a_{\lfloor \frac{r}{q} \rfloor + 1} + a_{k_{r+1}} \\ =& q \sum_{i=1}^{\lfloor \frac{r}{q} \rfloor} a_i + (r + 1 - q\lfloor \frac{r}{q} \rfloor) a_{\lfloor \frac{r}{q} \rfloor + 1} \\ \geq& q \sum_{i=1}^{\lfloor \frac{r}{q} \rfloor + 1} a_i + (r + 1 - q\lfloor \frac{r+1}{q} \rfloor) a_{\lfloor \frac{r+1}{q} \rfloor + 1},\end{aligned}$$

which means that (14) is satisfied for $r + 1$. Hence we established (14).
As B satisfies (1), we have

$$m = |B \cap A| \geq (1-p)n = qn,$$

which implies

$$n \leq \lfloor \frac{m}{q} \rfloor.$$

As (14) is satisfied for $r = m$, we have

$$\begin{aligned}||B|| = \sum_{i=1}^{m} a_{k_i} &\geq q(a_1 + a_2 + \ldots + a_{\lfloor \frac{m}{q} \rfloor}) \\ &\geq q(a_1 + a_2 + \ldots + a_n) \\ &= q||A|| = (1-p)||A||,\end{aligned}$$

as desired. ■

Proof of Lemma 3: Let $\alpha = \frac{1}{1+\log(\frac{m_1+T}{m_1})}$. As V is the capacity region under erasure-free case and $\alpha < 1$, $\alpha\vec{v}$ is achievable by the "as uniform as possible" intra-session coding under the erasure-free case.
Assume to the contrary that (5) is not satisfied for some $k \in \{1, 2, \ldots, n\}$. Without loss of generality, we may assume that k is the smallest such integer. Then we have

$$\sum_{i=1}^{k} q_j^i = 1, \quad \forall j : 1 \leq j \leq m_k - T. \quad (16)$$

Using (16), we get:

$$\alpha m_k \geq \sum_{i=1}^{k} \alpha v_i = \sum_{j=1}^{m_k} \sum_{i=1}^{k} q_j^i \geq \sum_{j=1}^{m_k - T} \sum_{i=1}^{k} q_j^i = m_k - T,$$

which implies:

$$m_k \leq \frac{T}{1-\alpha}. \quad (17)$$

Let's first prove that

$$m_k < m_1 + 2T. \quad (18)$$

Using (17), we get

$$m_k \leq \frac{T}{1-\alpha} = \frac{T}{1 - \frac{1}{1+\log(\frac{m_1+T}{m_1})}} = \frac{T(1 + \log(\frac{m_1+T}{m_1}))}{\log(\frac{m_1+T}{m_1})}.$$

Hence it is enough to show that

$$\frac{T(1 + \log(\frac{m_1+T}{m_1}))}{\log(\frac{m_1+T}{m_1})} < m_1 + 2T. \quad (19)$$

Let $x = \frac{T}{m_1}$. Then, (19) is equivalent to:

$$g(x) = (1 + \frac{1}{x}) \log(1 + x) > 1, \quad (20)$$

which follows immediately by the fact that $g'(x) > 0$ and $\lim_{x \to 0} g(x) = 1$.

Let t be the smallest integer satisfying $m_t \geq m_k - T + 1$. Hence,

$$m_{t-1} \leq m_k - T. \tag{21}$$

Let s be the smallest integer satisfying

$$\sum_{i=1}^{s+1} \frac{\alpha v_i}{m_i} \geq 1. \tag{22}$$

Let $X = v_1 + v_2 + \ldots + v_{t-1}$, $Y = v_1 + v_2 + \ldots + v_{s+1}$. Using (22) we get:

$$\begin{aligned}1 \leq \sum_{i=1}^{s+1} \frac{\alpha v_i}{m_i} &< \alpha + \sum_{i=1}^{\lfloor Y \rfloor - m_1} \frac{\alpha}{m_1 + i} + \frac{\alpha(Y - \lfloor Y \rfloor)}{Y} \\ &< \alpha \left(1 + \sum_{i=1}^{\lfloor Y \rfloor - m_1} \frac{1}{m_1 + i} + \frac{(Y - \lfloor Y \rfloor)}{Y}\right) \\ &< \alpha \left(1 + \log(\frac{Y}{m_1})\right).\end{aligned}$$

Hence,

$$Y > m_1 + T. \tag{23}$$

As $m_{s+1} \geq Y$, (18), (21) and (23) implies:

$$m_{s+1} \geq Y > m_1 + T \geq m_k - T + 1 > m_{t-1},$$

which implies:

$$s + 1 \geq t.$$

Let p_i denote the length of the uniform block for i-th receiver. Let $r = m_k - T + 1$. By assumption, $p_k < T$, which implies:

$$\sum_{i=t}^{k-1} q_r^i + \frac{\alpha v_k}{T} > 1. \tag{24}$$

As $p_i \geq T$ for $i \in \{1, 2, \ldots, k-1\}$, and $q_r^i \leq \alpha v_i / p_i$, (24) implies:

$$\begin{aligned}1 < \sum_{i=t}^{k-1} q_r^i + \frac{\alpha v_k}{T} &= \sum_{i=t}^{s} \frac{\alpha v_i}{m_i} + \sum_{i=s+1}^{k-1} q_r^i + \frac{\alpha v_k}{T} \\ &\leq \sum_{i=t}^{s} \frac{\alpha v_i}{m_i} + q_r^{s+1} + \sum_{i=s+2}^{k-1} \frac{\alpha v_i}{p_i} + \frac{\alpha v_k}{T} \\ &\leq \sum_{i=t}^{s} \frac{\alpha v_i}{m_i} + q_r^{s+1} + \sum_{i=s+2}^{k} \frac{\alpha v_i}{T}.\end{aligned}$$

Let $S = \sum_{i=t}^{s} \frac{\alpha v_i}{m_i} + q_r^{s+1} + \sum_{i=s+2}^{k} \frac{\alpha v_i}{T}$. Let's maximize S under the condition (22). As increasing v_{s+2} and decreasing v_{s+1} at the same amount increases S, we may assume that equality is satisfied in (22), i.e.

$$\sum_{i=1}^{s+1} \frac{v_i}{m_i} = \frac{1}{\alpha} = 1 + \log(\frac{m_1 + T}{m_1}). \tag{25}$$

Hence

$$S = \sum_{i=t}^{s+1} \frac{\alpha v_i}{m_i} + \sum_{i=s+2}^{k} \frac{\alpha v_i}{T} \geq 1. \tag{26}$$

Let

$$h(v_1, v_2, \ldots, v_n) = \sum_{i=t}^{s+1} \frac{v_i}{m_i} + \sum_{i=s+2}^{k} \frac{v_i}{T}.$$

Let's prove that

$$h(v_1, v_2, \ldots, v_n) \leq \frac{1}{\alpha} = 1 + \log(\frac{m_1 + T}{m_1}), \tag{27}$$

which will contradict (26). As

$$h(v_1, v_2, \ldots, v_n) \leq \sum_{i=t}^{s+1} \frac{v_i}{m_i} + \frac{m_k - Y}{T},$$

in order to verify (27), it is enough to show that

$$\sum_{i=t}^{s+1} \frac{v_i}{m_i} + \frac{m_k - Y}{T} \leq \frac{1}{\alpha}. \tag{28}$$

Using (25), (28) is equivalent to:

$$T \sum_{i=1}^{t-1} \frac{v_i}{m_i} + Y \geq m_k. \tag{29}$$

Let

$$\beta = \sum_{i=1}^{t-1} \frac{v_i}{m_i}.$$

Then, clearly

$$X = \sum_{i=1}^{t-1} v_i \geq m_1 \sum_{i=1}^{t-1} \frac{v_i}{m_i} = m_1 \beta, \tag{30}$$

$$Y - X = \sum_{i=t}^{s+1} v_i \geq m_t \sum_{i=t}^{s+1} \frac{v_i}{m_i} = m_t(\frac{1}{\alpha} - \beta). \tag{31}$$

We will consider two cases:
Case (1): $m_t \geq m_1 + T$.
Using (30) and (31), we get:

$$\begin{aligned}T \sum_{i=1}^{t-1} \frac{v_i}{m_i} + Y &= T\beta + X + Y - X \\ &\geq T\beta + m_1\beta + (\frac{1}{\alpha} - \beta) m_t \\ &\geq T\beta + m_1\beta + (\frac{1}{\alpha} - \beta)(m_1 + T) \\ &= \frac{m_1 + T}{\alpha} \geq m_1 + 2T,\end{aligned} \tag{32}$$

where the last inequality is equivalent to $\log(1+t) \geq \frac{t}{t+1}$ after setting $t = T/m_1$, hence follows from the inequality (20). As (32) implies (29), and (29) is equivalent to (27), we get a contradiction.

Case (2): $m_t < m_1 + T$.
If $\beta \geq 1$, as $Y \geq m_1 + T$, we have

$$T \sum_{i=1}^{t-1} \frac{v_i}{m_i} + Y = T\beta + Y \geq m_1 + 2T \geq m_k,$$

which establishes (29). Hence we get a contradiction.
We may assume that $\beta < 1$. Using (23), we have $Y > m_1 + T > m_t$. Hence

$$\frac{1}{\alpha} - \beta = \sum_{i=t}^{s+1} \frac{v_i}{m_i} \leq \frac{m_t - X}{m_t} + \log(\frac{Y}{m_t}),$$

which is equivalent to:

$$Y \geq m_t e^{\frac{1}{\alpha} - \beta + \frac{X}{m_t} - 1}. \tag{33}$$

Using (30) and (33), we get:

$$\begin{aligned} T \sum_{i=1}^{t-1} \frac{v_i}{m_i} + Y &= T\beta + Y \\ &\geq T\beta + m_t e^{\frac{1}{\alpha} - \beta + \frac{X}{m_t} - 1} \\ &\geq T\beta + m_t e^{\frac{1}{\alpha} - \beta + \frac{m_1 \beta}{m_t} - 1} \\ &= T\beta + m_t e^{1 + \log(1 + \frac{T}{m_1}) - \beta + \frac{m_1 \beta}{m_t} - 1} \\ &= T\beta + m_t (1 + \frac{T}{m_1}) e^{\beta(\frac{m_1}{m_t} - 1)}. \end{aligned} \tag{34}$$

Let $f(x) = Tx + m_t(1 + \frac{T}{m_1})e^{x(\frac{m_1}{m_t} - 1)}$. Then,

$$\begin{aligned} f'(x) &= T + m_t(1 + \frac{T}{m_1})(\frac{m_1}{m_t} - 1)e^{x(\frac{m_1}{m_t} - 1)} \\ &\geq T + m_t(1 + \frac{T}{m_1})(\frac{m_1}{m_t} - 1) \\ &= T + (1 + \frac{T}{m_1})(m_1 - m_t). \end{aligned} \tag{35}$$

If $T + (1 + \frac{T}{m_1})(m_1 - m_t) \leq 0$, using (34) and (35) we get:

$$\begin{aligned} & T \sum_{i=1}^{t-1} \frac{v_i}{m_i} + Y \\ & \geq f(\beta) \\ & \geq f(0) + \beta[T + (1 + \frac{T}{m_1})(m_1 - m_t)] \\ & = m_t(1 + \frac{T}{m_1}) + \beta[T + (1 + \frac{T}{m_1})(m_1 - m_t)] \\ & \geq m_t(1 + \frac{T}{m_1}) + [T + (1 + \frac{T}{m_1})(m_1 - m_t)] \\ & = m_1 + 2T \geq m_k, \end{aligned}$$

which establishes (29). Hence we get a contradiction.
If $T + (1 + \frac{T}{m_1})(m_1 - m_t) > 0$, using (34) and (35) we get:

$$\begin{aligned} T \sum_{i=1}^{t-1} \frac{v_i}{m_i} + Y \geq f(\beta) \geq f(0) = m_t(1 + \frac{T}{m_1}) \geq m_t + T \\ \geq m_k, \end{aligned}$$

which establishes (29), again we get a contradiction. ∎

Proof of Lemma 4: As $U \subset V$, if $p \leq \frac{1}{T}$, then clearly $U \subset (1 - p + \frac{1}{T})V$.
Let $p > \frac{1}{T}$. Define $q = 1 - p + \frac{1}{T}$.
Define $Z \subset \mathcal{I}$ as

$$|Z \cap \{l_1, l_2, \ldots, l_k\}| = \lfloor qk \rfloor \quad \forall k : 1 \leq k \leq m_k.$$

As $0 \leq q < 1$, Z is well-defined and unique. Let's prove that Z is a (p, T)-unerased subset of $\mathcal{I}$. Let $t \geq T$ and $0 \leq i \leq m_n - t$. We have:

$$\begin{aligned} |Z \cap \{l_{i+1}, a_{i+2}, \ldots, l_{i+t}\}| &= \lfloor q(i+t) \rfloor - \lfloor qi \rfloor \\ &> q(i+t) - qi - 1 \\ &= (1 - p + \frac{1}{T})t - 1 \\ &\geq (1-p)t, \end{aligned}$$

as desired.
Let $(u_1, u_2, \ldots u_n) \in U$. Applying cut-set bounds for Z, we have:

$$\sum_{i=1}^{k} u_i \leq |Z \cap \{l_1, l_2, \ldots, l_{m_k}\}| = \lfloor q m_k \rfloor \leq q m_k,$$

which implies that $U \subset qV = (1 - p + \frac{1}{T})V$, as desired. ∎

Proof of Lemma 5: Let $\vec{v} \in V$. Let $\alpha = \frac{1}{1 + \log(\frac{m_1 + T}{m_1})}$.
By Lemma 3, the rate vector $\alpha \vec{v}$ is achievable by the "as uniform as possible" intra-session coding in a such way that the corresponding allocation q satisfies:

$$q^i_{m_i - T + 1} = q^i_{m_i - T + 2} = \ldots = q^i_{m_i} \quad \forall i : 1 \leq i \leq n.$$

Let $Q_i = \{q^i_{m_i}, q^i_{m_i - 1}, \ldots, q^i_1\}$. We know that $q^i_{m_i} \geq q^i_{m_i - 1} \geq \ldots \geq q^i_1$. Let Q'_i be a (p, T)-unerased subset of Q_i. By Lemma 2:

$$||Q'_i|| \geq (1-p)||Q_i|| = (1-p)\alpha v_i.$$

Hence $(1-p)\alpha \vec{v} \in W$, which establishes (7). ∎

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