Amplify-and-Forward in Wireless Relay Networks

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***Abstract*—A general class of wireless relay networks with a single source-destination pair is considered. Intermediate nodes in the network employ an amplify-and-forward scheme to relay their input signals. In this case the overall input-output channel from the source via the relays to the destination effectively behaves as an intersymbol interference channel with colored noise. Unlike the previous work, we formulate the problem of the maximum achievable rate in this setting as an optimization problem with no assumption on the network size, topology, and received signal-to-noise ratio. Previous work considered only scenarios wherein relays use all their power to amplify their received signals. We demonstrate that this may not always maximize the maximal achievable rate in amplify-and-forward relay networks. The proposed formulation allows us to not only recover known results on the performance of the amplify-and-forward schemes for some simple relay networks but also characterize the performance of more complex amplify-and-forward relay networks which cannot be addressed in a straightforward manner using existing approaches.**

Using cut-set arguments, we derive simple upper bounds on the capacity of general wireless relay networks. Through various examples, we show that a large class of amplify-and-forward relay networks can achieve rates within a constant factor of these upper-bounds asymptotically in network parameters.

I. Introduction

Since its introduction in [1], *Amplify-and-Forward (AF)* relay schemes have been studied in the context of *cooperative communication* [2], [3], estimating the capacity of relay networks [4]–[6], and *analog network coding* [7]–[11]. For cooperative communication, AF schemes provide spatial diversity to fight against fading; for capacity estimation of relay networks, such schemes provide achievable lower bounds that are known to be optimal in some communication scenarios; and for analog network coding, given the broadcast nature of the wireless medium that allows the *mixing* of the signals in the air, these schemes provide a communication strategy that achieves high throughput with low computational complexity at internal nodes. In this paper, we concern ourselves mostly with the capacity analysis of a general class of Gaussian AF relay networks. Extension of our method and results to cooperative communication and analog network coding scenarios is the part of our future work.

In the previous work, while analyzing the performance of AF schemes in relay networks one or more of the following assumptions have been made: networks with a small number of nodes [8], [10]; networks with simple topologies [3]–[5], [8], [10]; or relay operation in the high-SNR regime, [10]. However, for two reasons, we believe that it is important to characterize the performance of the AF schemes without such assumptions. First, we feel that for a scheme such as amplify-and-forward that allows us to exploit the broadcast nature of the wireless medium such assumptions on the size and topology may result in lower achievable performance than otherwise. Second, even in the low-SNR regimes amplify-and-forward can be capacity-achieving relay strategy in some scenarios, [5]. Therefore, a framework to address the performance of AF schemes in general wireless relay networks is desired.

However, one major issue with constructing such a framework is the following. In general wireless relay networks with AF relaying, the resulting input-output channel between the source and the destination is an intersymbol interference (ISI) channel ([4], [10]) with colored noise. This is so because both the source signal and the noise introduced at the relay nodes may reach the destination via multiple paths with differing delays. This results in a formidable problem to analyze with the existing methods without the assumptions above.

Our main contribution is that we provide such a framework to compute the maximum achievable rate with AF schemes for a class of general wireless relay networks, namely Gaussian relay networks. This framework casts the problem of computing the maximum rate achievable with AF relay networks as an optimization problem. We emphasize that our work shows that amplifying the received signal to the maximum possible value at intermediate nodes might result in sub-optimal end-to-end throughput. Also, we establish the generality of the proposed formulation by showing that it allows us to derive in a unified and simple manner not only the various existing results on the performance of simple AF relay networks but also new results for more complex networks that cannot be addressed in a straightforward manner with existing methods. We show through various examples that for a large class of relay networks the AF schemes can achieve rates within a constant factor of the cut-set upper-bounds on the capacity of general wireless relay networks.

In this paper, we provide the summary of our work. We have omitted most proofs or give only brief outlines. The proofs and discussions can be found in our arXiv submission [14].

Organization: In Section II we introduce the general class of Gaussian AF relay networks addressed in this paper. In Section III we formulate the problem of maximum rate achievable via AF schemes in these networks. In Section IV we compute the rates achievable via AF schemes for two instances of such relay networks under various communication scenarios, and then in Section V we discuss the asymptotic behavior of the gap between these rates and the corresponding upper bounds on the capacity of general wireless relay networks computed there. Section VI concludes the paper with a summary.

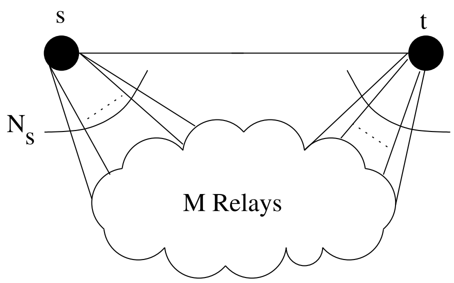

Fig. 1. A single source-single destination communication channel over general Gaussian relay network with M relays.

II. SYSTEM MODEL

Let us consider a $(M+2)$-node wireless relay network with source s destination t and M relays as a directed graph $G=(V,E)$ with bidirectional links, as shown in Figure 1. Each node in the network is assumed to have a single antenna. Let us assume that the degree of the source node is N_s+1, with it being connected to the destination node and a subset S_s of the relay nodes, $N_s=|S_s|$. Similarly, let us assume that the degree of the destination node is N_t+1, with it being connected to the source node and a subset S_t of the relay nodes, $N_t=|S_s|$. In general, $S_s\cup S_t\subseteq V\setminus\{s,t\}$.

At instant n, the channel output at node $i, i\in V\setminus\{s\}$, is

$$y_i[n]=\sum_{j\in\mathcal{N}(i)} h_{ji}x_j[n]+z_i[n], \quad -\infty<n<\infty, \quad (1)$$

where $x_j[n]$ is the channel input of the node j in the neighbor set $\mathcal{N}(i)$ of node i. In (1), h_{ji} is a real number representing the channel gain along the link from relay j to relay i. It is assumed to be fixed (for example, as in a single realization of a fading process) and known throughout the network. Further, $\{z_i[n]\}$ is a sequence (in n) of independently and identically distributed (*i.i.d.*) Gaussian random variables with zero mean and variance $\sigma^2, z_i[n]\sim\mathcal{N}(0,\sigma^2)$. We also assume that z_i are independent of the input signal and of each other. The source symbols $x_s[n], -\infty<n<\infty$, are *i.i.d.* Gaussian random variables with zero mean and variance P_s that satisfies average source power constraint, $x_s[n]\sim\mathcal{N}(0,P_s)$. We assume that the i^{th} relay node's transmit power is constrained as:

$$E[x_i^2[n]]\le P_i, \quad -\infty<n<\infty \quad (2)$$

Motivated by some work on analog network coding, such as [15], let us modify the relay operation in Section II as follows.

Relay operation: We assume that each relay node maintains a buffer of signals it forwarded previously. Therefore, each relay node i, after receiving the channel output $y_i[n]$ at time instant n executes the following series of steps:

Step 1: Obtain the *residual signal* $y_i'[n]$ by subtracting the contributions of previously forwarded signals from $y_i[n]$.

Step 2: Compute the power $P'_{R,i}$ of the *residual signal* $y_i'[n]$.

Step 3: At instant $n+1$ transmit the scaled version of the *residual signal* $y_i'[n]$ of its input at time instant n:

$$x_i[n+1]=\beta_i y_i'[n], \quad 0\le\beta_i^2\le\beta_{i,max}^2=P_i/P'_{R,i}, \quad (3)$$

where the β_i is the scaling factor[1]. Let the network-wide amplification vector for the M relay nodes be denoted as $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_M)$.

One of the major advantages of this relay operation is that by subtracting the previously forwarded signals from their inputs, the relays expend their transmit power in forwarding only the "new" information.

Using (1) and (3), the input-output channel between the source and destination can be written as an intersymbol interference (ISI) channel that at instant n is given by

$$\begin{aligned} & y_t[n] \quad (4)\\ &= h_{st}x_s[n]+z_t[n]\\ &+\sum_{d=1}^{D^s}\Bigg[\sum_{(i_1,\ldots,i_d)\in K_d} h_{si_1}\beta_{i_1}h_{i_1i_2}\ldots h_{i_{d-1}i_d}\beta_{i_d}h_{i_dt}\Bigg]x_s[n-d]\\ &+\sum_{d=1}^{D^1}\Bigg[\sum_{(i_1,\ldots,i_d)\in K_{1,d}} \beta_1 h_{1i_1}\ldots h_{i_{d-1}i_d}\beta_{i_d}h_{i_dt}\Bigg]z_1[n-d]\\ &\quad\vdots\\ &+\sum_{d=1}^{D^M}\Bigg[\sum_{(i_1,\ldots,i_d)\in K_{M,d}} \beta_M h_{Mi_1}\ldots h_{i_{d-1}i_d}\beta_{i_d}h_{i_dt}\Bigg]z_M[n-d], \end{aligned}$$

where $K_d, 1\le d\le D^s$, is the set of d-tuples of node indices corresponding to all paths from the source to the destination with path delay d and D^s is the delay of the longest such path. Note that along such paths $D^s\le M$. Similarly, $K_{m,d}, 1\le m\le M, 1\le d\le D^m$, is the set of d-tuples of node indices corresponding to all paths from the m^{th} relay to the destination with path delay d, D^m is the delay of the longest such path from m^{th} relay to the destination. It should be noted that $\max(D^1,\ldots,D^M)=D^s-1$.

Let us introduce *modified* channel parameters as follows. For all the paths between the source and the destination:

$$\begin{aligned} h_0 &= h_{st} \quad (5)\\ h_d &= \sum_{(i_1,\ldots,i_d)\in K_d} h_{si_1}\beta_{i_1}h_{i_1i_2}\ldots h_{i_{d-1}i_d}\beta_{i_d}h_{i_dt}, 1\le d\le D^s \end{aligned}$$

For all the paths between the m^{th}-relay, $1\le m\le M$, and the destination:

$$\begin{aligned} h_{m,0} &= 0,\\ h_{m,d} &= \sum_{(i_1,\ldots,i_d)\in K_{m,d}} \beta_m h_{mi_1}\ldots h_{i_{d-1}i_d}\beta_{i_d}h_{i_dt}, 1\le d\le D^m \end{aligned}$$

In terms of these modified channel parameters, the source-destination ISI channel in (4) can be written as:

$$\begin{aligned} y_t[n] &= \sum_{j=0}^{D^s} h_j x_s[n-j]+\sum_{j=0}^{D^1} h_{1,j}z_1[n-j] \quad (6)\\ &\quad +\ldots+\sum_{j=0}^{D^M} h_{M,j}z_M[n-j]+z_t[n] \end{aligned}$$

[1]Note that, in general, β_i may depend on i^{th} relay's past observations. $\beta_i[n]=f_{i,n}(Y_i[n-1],\ldots,Y_i[1])$. However, due to practical considerations, such as low-delay operation, we do not consider such scenarios here.

III. ACHIEVABLE RATE FOR THE SOURCE-DESTINATION ISI CHANNEL IN AF RELAY NETWORKS

Lemma 1: For a given $\boldsymbol{\beta}$, the achievable rate for the channel in (6) with *i.i.d.* Gaussian input is:

$$I(P_s, \boldsymbol{\beta}) = \frac{1}{2\pi} \int_0^{\pi} \log \left[1 + \frac{P_s}{\sigma^2} \frac{|H(\lambda)|^2}{1 + \sum_{m=1}^{M} |H_m(\lambda)|^2}\right] d\lambda, \quad (7)$$

where

$$H(\lambda) = \sum_{j=0}^{D^s} h_j e^{-ij\lambda}, H_m(\lambda) = \sum_{j=0}^{D^m} h_{m,j} e^{-ij\lambda}, i = \sqrt{-1} \quad (8)$$

Proof: In [12] a *Discrete Fourier Transform (DFT)* based formalism is developed to compute the capacity of the Gaussian channel with ISI. We compute the maximum achievable rate for the channel in (6) for a given $\boldsymbol{\beta}$ by generalizing this formalism to also include the ISI channel for the Gaussian noise at each relay node resulting in colored Gaussian noise at the destination. The details of the proof are in [14]. ∎

For a given network-wide amplification vector $\boldsymbol{\beta}$, the achievable information rate is given by $I(P_s, \boldsymbol{\beta})$. Therefore, the maximum information-rate $I_{AF}(P_s)$ achievable in an AF relay network with *i.i.d.* Gaussian input is defined as the maximum of $I(P_s, \boldsymbol{\beta})$ over all feasible $\boldsymbol{\beta}$, subject to per relay-node amplification constraint (3). In other words:

$$\text{(P1):} \qquad I_{AF}(P_s) \stackrel{def}{=} \max_{\boldsymbol{\beta}: 0 \le \beta_i^2 \le \beta_{i,max}^2} I(P_s, \boldsymbol{\beta}) \quad (9)$$

The problem P1 implies that in general AF relay networks, if the relays amplify the received signals to the maximum possible then it may result in sub-optimal end-to-end throughput. It is different from all the previous work where the relays use all their power to amplify the received signal. The following example illustrates the significance of this observation.

Example 1: Let us consider the well-studied diamond network, [10]. It is defined as $G = (V, E)$ with $V = \{s, t, 1, 2\}$ and $E = \{(s,1), (s,2), (1,t), (2,t)\}$. Let $h_{s1} = 1, h_{s2} = 0.1, h_{1t} = h_{2t} = 1$. Let $P_s = P_1 = P_2 = 10$ and noise variance $\sigma^2 = 0.1$ at each node. Therefore, we have

$$\beta_{1,max}^2 = \frac{P_1}{h_{s1}^2 P_s + \sigma^2} = 0.99, \ \beta_{2,max}^2 = \frac{P_2}{h_{s2}^2 P_s + \sigma^2} = 50.0$$

From (9), we have the following rate maximization problem:

$$I_{AF} = \max_{\beta_1, \beta_2} \frac{1}{2} \log \left[1 + 100 \frac{(\beta_1 + 0.1\beta_2)^2}{1 + \beta_1^2 + \beta_2^2}\right]$$

subject to constraints $0 \le \beta_1^2 \le \beta_{1,max}^2$ and $0 \le \beta_2^2 \le \beta_{2,max}^2$.

The objective function is plotted in the Figure 2 for $\beta_1 = 0.995$. The optimal solution of this problem is $(\beta_1 = 0.995, \beta_2 = 0.225)$. Therefore, it follows that in this case $\beta_2 = \beta_{2,max}$ is not the optimal amplification factor. ∎

With this observation and the definition of the relay operation given in the previous section, it is appropriate to call the forwarding scheme proposed in this paper as *subtract-scale-and-forward*.

The problem P1 is computationally-hard to solve for all but some trivial relay networks [14]. However under some assumptions on relay operations, its solution can be efficiently approximated. This we discuss in detail in [14] and demonstrate below for two instances of general AF relay networks.

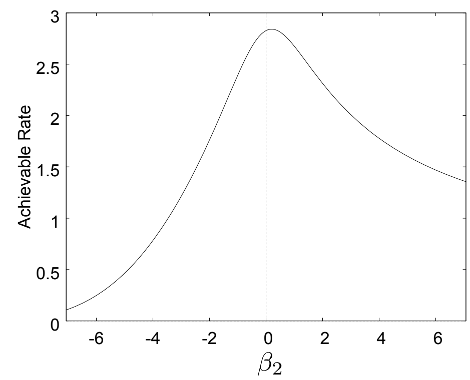

Fig. 2. The achievable rate for the Example 1 when $\beta_1 = \beta_{1,max}$ and β_2 lies in $[-\beta_{2,max}, \beta_{2,max}]$.

IV. APPROXIMATING THE RATE $I_{AF}(P_s)$

Let us consider the problem P1 when the relays operate instantaneously, that is the relays amplify-and-forward their input signals without delay, [8]. Therefore, we have $x_i[n] = \beta_i y_i'[n], 1 \le i \le M$. Note the possible system instability resulting from this assumption is avoided by the modified relay-operation discussed above. With this assumption, (P1) reduces to (after setting $\lambda = 0$ in (9) and then integrating):

$$\text{(P2):} \ I_{AF}(P_s) = \max_{\boldsymbol{\beta}} \frac{1}{2} \log \left[1 + \frac{P_s}{\sigma^2} \frac{|H(0)|^2}{1 + \sum_{m=1}^{M} |H_m(0)|^2}\right] \quad (10)$$

such that $0 \le \beta_i^2 \le \beta_{i,max}^2 = P_i / P'_{R,i}$.

Let us consider the problem P2 when the relay nodes are constrained to use the same amplification-factor, that is, $\beta_i = \beta$, for all $1 \le i \le M$. In the practical setting, this assumption considerably simplifies the system-design with β set to one particular value for all relay nodes. Then, (P2) reduces to

$$\text{(P3):} \ I_{AF}(P_s) = \max_{0 \le \beta^2 \le \beta_{max}^2} \log \left[1 + \frac{P_s}{\sigma^2} \frac{|H(0)|^2}{1 + \sum_1^M |H_m(0)|^2}\right] \quad (11)$$

Note that the solution of (P2) cannot be smaller than the solution of (P3) because the set of feasible $\boldsymbol{\beta} : \beta_i = \beta$, for (P3) is a subset of the set of feasible $\boldsymbol{\beta}$ for (P2).

In general, $\beta_{i,max}^2 \sim P_i$. Therefore $\sum \beta_{i,max}^2 \sim \sum P_i$. However, as we are considering the case of equal β, $\beta_{i,max} = \beta_{max}$, so we have $M\beta_{max}^2 \sim \sum P_i$ or $\beta_{max}^2 \sim M^{-1} \sum P_i$.

Next, we discuss the solution of (P3) in different scenarios for two special cases of the general class of relay networks we address in this paper.

A. Type-A Relay Network

Definition: For one source-destination pair and M relays, Type A network is defined as: $G = (V, E)$, where $V = \{s, t, 1, \ldots, M\}$ and $E = \{(s,t), (s,i), (i,t) : i \in \{1, \ldots, M\}\}$, as illustrated in Figure 3.

We solve the problem P3 in the following two scenarios.

Scenario 1 (No attenuation network): Let us assume that there is no attenuation along any link in the network, that is,

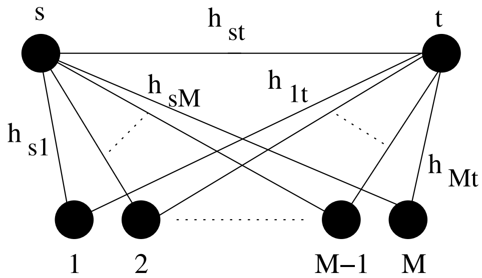

Fig. 3. Type A Relay Network.

$h_{st} = h_{si} = h_{it} = 1$ for all $1 \leq i \leq M$. The problem P3 in this case is:

***Proposition** 1:*

$$I_{AF}(P_s) = \max_{0\leq\beta^2\leq\beta^2_{max}} \frac{1}{2}\log\left[1 + \frac{P_s}{\sigma^2}\frac{(1+M\beta)^2}{1+M\beta^2}\right]$$

***Lemma** 2:* $I_{AF}(P_s)$ attains global maximum at $\beta_{opt} = 1$.

Now let us consider two particular ways in which β_{max} varies with network size.

Scenario 1, Case A (Increasing relay power): For the given network with M relay nodes, let us assume that the sum power of the relay nodes is constrained as follows:

$$\sum_{m=1}^{M} E[X_m^2] \leq \sum_{m=1}^{M} P_i \leq M^{u+1}Q, u > 0, Q = constant$$

So $\beta^2_{max} = M^u Q$. From Lemma 2, we have for $M \rightarrow \infty$.

$$I_{AF}(P_s) = \begin{cases} \frac{1}{2}\log[1 + \frac{P_s}{\sigma^2}(1+M)], \text{ if } \beta_{max} \geq 1, \\ \frac{1}{2}\log[1 + \frac{P_s}{\sigma^2}(M + \frac{2}{\beta_{max}})], \text{ otherwise} \end{cases} \quad (12)$$

Scenario 1, Case B (Constant total relay power): Let us consider the case where the sum power of relay nodes is fixed irrespective of the number of relay nodes, that is $\sum_{m=1}^{M} P_i \leq Q, Q = constant$. Therefore, we set $\beta^2_{max} = \frac{Q}{M}$. As $M \rightarrow \infty$, for sufficiently large M, $\beta_{max} < \beta_{opt} = 1$. Therefore, from Lemma 2, $\beta = \beta_{max}$ maximizes the achievable rate and we have for $M \rightarrow \infty$

$$I_{AF}(P_s) = \frac{1}{2}\log\left[1 + \frac{P_s}{\sigma^2}\frac{Q}{1+Q}M\right] \quad (13)$$

Scenario 2 (Bounded channel gains): Let us consider the scenario where the channel gains are arbitrary, but strictly bounded, $0 < h_{st}, h_{si}, h_{it} < \infty$, $1 \leq i \leq M$. The problem P3 in this case is:

***Proposition** 2:*

$$I_{AF}(P_s) = \max_{0\leq\beta^2\leq\beta^2_{max}} \frac{1}{2}\log\left[1+\frac{P_s}{\sigma^2}\frac{(h_{st} + \beta\sum_{i=1}^{M} h_{si}h_{it})^2}{1+\beta^2\sum_{i=1}^{M} h_{it}^2}\right]$$

***Lemma** 3:* $I_{AF}(P_s)$ attains global maximum at $\beta_{opt} = \frac{\sum_{i=1}^{M} h_{si}h_{it}}{h_{st}\sum_{i=1}^{M} h_{it}^2}$.

Increasing relay power: Let us consider the increasing total relay power scenario as in Scenario 1, Case A. Let $\beta^2_{max} = M^u Q$. In this case, following Lemma 3, we obtain the following lower bound on the achievable rate as $M \rightarrow \infty$:

$$\begin{aligned} I_{AF}(P_s) &> \frac{1}{2}\log\left[1 + \frac{P_s}{\sigma^2}(Mh_{min}+1)\frac{h^2_{s,max}h_{min}}{h_{max}}\right], \quad (14) \\ &\quad \text{if } \beta_{max} \geq \frac{\sum_{i=1}^{M} h_{si}h_{it}}{h_{st}\sum_{i=1}^{M} h_{it}^2}, \\ I_{AF}(P_s) &> \frac{1}{2}\log\left[1 + \frac{P_s}{\sigma^2}(Mh_{min} + \frac{1}{\beta_{max}})\frac{h^2_{s,max}h_{min}}{h_{max}}\right], \\ &\quad \text{otherwise,} \end{aligned}$$

where $h_{s,max} = \max\{h_{st}, h_{s1}, \ldots, h_{sM}\}$, $h_{min} = \min\{h_{1t}, \ldots, h_{Mt}\}$, and $h_{max} = \max\{h_{1t}, \ldots, h_{Mt}\}$.

B. Type-B Relay Network

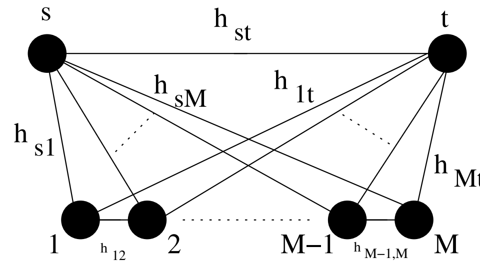

Fig. 4. Type B Relay Network.

Definition: For one source-destination pair and M relays, Type B network is defined as: $G = (V, E)$, where $V = \{s, t, 1, \ldots, M\}$ and $E = \{(s,t), (s,i), (i,t), (j, j+1) : i \in \{1, \ldots, M\}, j \in \{1, \ldots, M-1\}\}$, as shown in Figure 4.

For such networks, we consider the no-attenuation scenario where all channel gains are set to unity, that is, $h_{st} = h_{si} = h_{it} = 1$, $1 \leq i \leq M$ as well as $h_{i,i+1} = 1$, $1 \leq i \leq M-1$. The problem P3 in this case is:

***Proposition** 3:*

$$\begin{aligned} I_{AF}(P_s, \beta) &= \frac{1}{2}\log\left[1 + \frac{P_s}{\sigma^2}\frac{\left(1 + \beta M\frac{\beta^M-1}{\beta-1} - \frac{M\beta^{M+1}}{\beta-1} + \beta^2\frac{\beta^M-1}{(\beta-1)^2}\right)^2}{1+\sum_{i=1}^{M}\beta^2\left(\frac{\beta^{M-(i-1)}-1}{\beta-1}\right)^2}\right] \\ I_{AF}(P_s) &= \max_{0\leq\beta^2\leq\beta^2_{max}} I_{AF}(P_s, \beta) \quad (15) \end{aligned}$$

It can be proved that the objective function is quasiconcave [13], therefore a unique global maximum exists. Let β that solves (15) be denoted as $\beta = \beta_{opt}$. However, obtaining a closed-form expression for β_{opt} does not appear straightforward, though it can be numerically computed for any M.

Constant total relay power: Let us consider the case where the sum power of relay nodes is fixed irrespective of the number of relay nodes as in Scenario 1, Case B. Let $\beta^2_{max} = \frac{Q}{M}$. As $M \rightarrow \infty$, for sufficiently large M, $\beta_{max} < \beta_{opt}$. Therefore, $\beta = \beta_{max}$ maximizes the achievable rate and we have the following rate achievable asymptotically as $M \rightarrow \infty$

$$I_{AF}(P_s) = \frac{1}{2}\log\left[1 + \frac{P_s}{\sigma^2}\frac{QM}{1+Q}\frac{1+\sqrt{Q/M}}{1-\sqrt{Q/M}}\right] \quad (16)$$

V. Asymptotic Capacity

We first derive an upper bound to the capacity of general wireless relay networks we address in this paper. We then discuss the asymptotic behavior of the gap between this upper bound and the lower bounds computed in the previous section.

***Proposition** 4:* The capacity C of a general wireless relay network is upper-bounded as $C \leq \min\{C_{BC}, C_{MAC}\}$, where C_{BC} and C_{MAC} are the upper bounds on the capacity of the broadcast cut at the source and multiple-access cut at the destination respectively, and are given as follows

$$\begin{aligned} C_{BC} &= \log\left[1 + \frac{P_s}{\sigma^2}(h_{st}^2 + \sum_{i \in S_s} h_{si}^2)\right] \\ C_{MAC} &= \log\left[1 + \frac{P_s + \sum_{i \in S_t} P_i}{\sigma^2}(h_{st}^2 + \sum_{i \in S_t} h_{it}^2)\right] \end{aligned}$$

Remark 3: The Proposition 3 in [4] can be obtained as a special case of this proposition by setting $N_s = N_t = M$.

A. Type A Relay Networks

No attenuation, increasing relay power: In this case, the broadcast bound C_{BC} is always asymptotically smaller than the multiple-access bound C_{MAC}, as follows from

$$2^{C_{MAC} - C_{BC}} \approx 1 + \frac{M^{u+1}Q}{P}, \text{ for large } M.$$

Therefore it suffices to compute the asymptotic gap between C_{BC} and the lower bound in (12). In fact, in this case we have

$$C_{BC} - 2I_{AF}(P_s) = 0, \text{ for all } M \geq 1$$

The actual capacity C of the relay network in this case is bounded by $C_{BC}/2 \leq C \leq C_{BC}$.

No attenuation, constant total relay power: In this case also the broadcast bound C_{BC} is asymptotically smaller than the multiple access-bound C_{MAC}, as shown below

$$\lim_{M \to \infty} 2^{C_{MAC} - C_{BC}} = 1 + \frac{Q}{P}$$

Therefore we only address the asymptotic gap between C_{BC} and the lower bound in (13). We have

$$\lim_{M \to \infty} C_{BC} - 2I_{AF}(P_s) = \frac{1}{2}\log(1 + 1/Q)$$

The actual capacity C of the relay network in this case is bounded by $\frac{1}{2}(C_{BC} - \frac{1}{2}\log(1 + 1/Q)) \leq C \leq C_{BC}$ and the bound gets tighter with increasing Q.

Bounded channel gains: The gap between C_{BC} and lower bound of achievable rate in (14) is bounded asymptotically as:

$$\lim_{M \to \infty} C_{BC} - 2I_{AF}(P_s) \leq \frac{1}{2}\log\left[\frac{h_{max}}{h_{min}^2}\right]$$

Note that the apparent looseness of the gaps computed above compared to the corresponding gaps in [4] arises from the series of simplifications made to reduce (P1) to (P3).

B. Type B Relay Networks

No attenuation, constant total relay power: As claimed above, the upper-bound in Proposition 4 holds for Type B networks too. Therefore we only address the asymptotic gap between C_{BC} and the lower bound in (16). We have

$$\lim_{M \to \infty} C_{BC} - 2I_{AF}(P_s) = \frac{1}{2}\log(1 + 1/Q)$$

The actual capacity C of the relay network in this case too is bounded by $\frac{1}{2}(C_{BC} - \frac{1}{2}\log(1 + 1/Q)) \leq C \leq C_{BC}$ and bound gets tighter with increasing Q.

VI. Conclusion and Future Work

We provide a framework to analyze the performance of the AF relay schemes in a general class of Gaussian relay networks. We show that compared to the existing methods, the proposed framework not only allows us to characterize the performance of general AF relay networks in a unified manner but it also allows for tighter characterization. We also show that AF schemes can be capacity achieving for a large class of wireless relay networks. An extension of our work also facilitates the computation of achievable rates for analog network coding scenarios for non-layered networks and low to moderate SNR regimes. We plan to address it in detail in our future work.

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