Analog Network Coding in General SNR Regime

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***Abstract*—The problem of maximum rate achievable with analog network coding for a unicast communication over a layered relay network with directed links is considered. A relay node performing analog network coding scales and forwards the signals received at its input. Recently this problem has been considered with two conditions: (1) each relay node scales its received signal to the maximum extent possible subject to its transmit power constraint, (2) the relay nodes in specific subsets of the network operate in the high-SNR regime. We establish that the condition (1), in general, leads to suboptimal end-to-end rate. We also prove that the performance of analog network coding in a wide class of layered networks without needing condition (2).**

The key contribution of this work is a lemma that states that the globally optimal set of scaling factors, which maximize end-to-end rate, for the nodes in a layered relay network can be computed layer-by-layer. Specifically, the rate-optimal set of scaling factors for the nodes in a layer is the one that maximizes the *sum-rate* of the nodes in the next layer. This result allows us to analyze analog network coding performance in even those network scenarios that cannot be addressed using the existing approaches. We illustrate this by computing the maximum rate achievable with analog network coding in one particular layered network, in various communication scenarios.

I. Introduction

Analog network coding (ANC) extends to multihop wireless networks the idea of linear network coding [1] where an intermediate node sends out a linear combination of its incoming packets. In a wireless network, signals transmitted simultaneously by multiple sources add in the air. Each node receives a *noisy sum* of these signals, *i.e.* a linear combination of the received signals and noise. A multihop relay scheme where an intermediate relay node merely amplifies and forwards this noisy sum is referred to as analog network coding [2], [3].

In [3], [4], the performance of the analog network coding in layered relay networks is analyzed in the high-SNR regime, that is, $\min_{k \in l} P_{R,k} \geq 1/\delta, l = 1, \dots, L$ for some small $\delta \geq 0$, where $P_{R,k}$ is the received signal power at the k^{th} node and L is the number layers of relay nodes. In [3], the achievable rate is computed under two assumptions. First, the nodes in all layers operate in the high-SNR regime (Assumption A), and second, each relay node amplifies the received signal to the maximum extent possible subject to its transmit power constraint (Assumption B). It is shown that the rate achieved under these two assumptions approaches network capacity as the source power increases. The authors in [4] extend this work to the scenarios where the nodes in at most one layer do not satisfy these two assumptions and show that achievable rates in such scenarios also approach the network capacity as the source power increases.

However, requiring each relay node to amplify its received signal to the upper bound of its transmit power constraint results in suboptimal end-to-end performance of analog network coding, as we establish in this paper and also previously indicated in [6]. Further, even in low-SNR regimes amplify-and-forward relaying can be capacity achieving relay strategy in some scenarios, [5]. Therefore, in this paper we are concerned with analyzing the performance of analog network coding in layered networks, without above two assumptions on received signal scaling factors and received SNRs. Computing the maximum rate achievable with analog network coding without these two assumptions, however results in a computationally intractable problem, in general [4], [6].

The main contribution of this paper is a result that states that the globally optimal set of scaling factors for the nodes that maximize end-to-end rate in a general layered relay network can be computed layer-by-layer. In particular, the rate-optimal set of scaling factors for the nodes in a layer is the one that maximizes the sum-rate of the nodes in the next layer. This result allows us to exactly compute the maximum rate achievable with analog network coding in a wider class of layered networks that cannot be so addressed using existing approaches. Further, for general layered relay networks, our result significantly reduces the computational complexity of solving this problem. We illustrate the significance of our result by computing the maximum ANC rate in different scenarios for one particular layered network.

In this paper, we provide the summary of our work. We have omitted most proofs or give only brief outlines. The details can be found in our arXiv submission [7].

Organization: In Section II we introduce a general wireless layered relay network model and formulate the problem of maximum rate achievable with ANC in such a network. Section III discusses the computational hardness of this problem and existing approaches to address it. In Section IV we first motivate and then state and prove the key lemma of this paper that allows us to compute the optimal set of scaling factors for the nodes in the network in a layer-by-layer manner. Then Section V illustrates the computation of the maximum ANC rate in one particular layered network in various scenarios. Finally, Section VI concludes the paper with a summary.

II. System Model

The layered network model in this paper follows the one proposed in [3].

Let us consider a $(M+2)$-node wireless relay network with the source s, the destination t and the set R of relay nodes as a directed graph $G = (V, E), V = \{s, t, R\}, |R| = M$ with unidirectional links. The nodes in the network are arranged in layers with the source s in layer '0', the destination t in layer '$L+1$', and the relay nodes in R arranged in L layers. The l^{th} layer contains n_l relay nodes, $\sum_{l=1}^{L} n_l = M$. An instance of such a network is given Figure 2. Each node is assumed to have a single antenna and operate in full-duplex mode.

At instant n, the channel output at node $i, i \in V \setminus \{s\}$, is

$$y_i[n] = \sum_{j \in \mathcal{N}(i)} h_{ji} x_j[n] + z_i[n], \quad -\infty < n < \infty, \tag{1}$$

where $x_j[n]$ is the channel input of the node j in the neighbor set $\mathcal{N}(i)$ of node i. In (1), h_{ji} is a real number representing the channel gain along the link from relay j to relay i. It is assumed to be fixed (for example, as in a single realization of a fading process) and known throughout the network. The source symbols $x_s[n], -\infty < n < \infty$, are *i.i.d.* Gaussian random variables with zero mean and variance P_s that satisfies an average source power constraint, $x_s[n] \sim \mathcal{N}(0, P_s)$. Further, $\{z_i[n]\}$ is a sequence (in n) of *i.i.d.* Gaussian random variables with $z_i[n] \sim \mathcal{N}(0, \sigma^2)$. We also assume that z_i are independent of the input signal and of each other. We assume that the i^{th} relay's transmit power is constrained as:

$$E[x_i^2[n]] \leq P_i, \quad -\infty < n < \infty \tag{2}$$

In analog network coding each relay node amplifies and forwards the noisy signal sum received at its input. More precisely, a relay node i at instant $n+1$ transmits the scaled version of $y_i[n]$, its input at time instant n, as follows

$$x_i[n+1] = \beta_i y_i[n], \quad 0 \leq \beta_i^2 \leq \beta_{i,max}^2 = P_i / P_{R,i}, \tag{3}$$

where $P_{R,i}$ is the received power at the node i and choice of the scaling factor β_i satisfies the power constraint (2).

In layered networks, all copies of a source signal and a noise symbol introduced at a node traveling along different paths arrive at the destination with same respective time delays. Therefore, the outputs of the source-destination channel are free of intersymbol interference. This simplifies the relation between input and output of the channel and allows us to omit the time-index while denoting the input and output signals.

Using (1) and (3), the input-output channel between the source and destination can be written as

$$\begin{aligned} y_t &= \Bigg[\sum_{(i_1,\ldots,i_L) \in K_L} h_{si_1} \beta_{i_1} h_{i_1 i_2} \ldots \beta_{i_L} h_{i_L t} \Bigg] x_s \\ &+ \sum_{l=1}^{L} \sum_{j=1}^{n_l} \Bigg[\sum_{(i_l,\ldots,i_L) \in K_{lj,L}} \beta_{lj} h_{lj,i_l} \ldots \beta_{Li_L} h_{Li_L,t} \Bigg] z_{lj} + z_t, \end{aligned} \tag{4}$$

where K_L, is the set of L-tuples of node indices corresponding to all paths from the source to the destination with path delay L. Similarly, $K_{lj,L-l+1}$, is the set of $L-l+1$-tuples of node indices corresponding to all paths from the j^{th} relay of l^{th} layer to the destination with path delay $L-l+1$.

For all the paths between the source s and the destination t, and all the paths between the j^{th} relay of l^{th} layer to the destination t with path delay $L-l+1$, we introduce *modified* channel gains, respectively, as follows

$$h_s = \sum_{(i_1,\ldots,i_L) \in K_L} h_{si_1} \beta_{i_1} h_{i_1 i_2} \ldots \beta_{i_L} h_{i_L t} \tag{5}$$

$$h_{lj} = \sum_{(i_1,\ldots,i_{L-l+1}) \in K_{lj,L-l+1}} \beta_{lj} h_{lj,i_1} \ldots \beta_{Li_L} h_{Li_L,t} \tag{6}$$

In terms of these modified channel gains[1], the source-destination channel in (4) can be written as:

$$y_t = h_s x_s + \sum_{l=1}^{L} \sum_{j=1}^{n_l} h_{lj} z_{lj} + z_t \tag{7}$$

Problem Formulation: For a given network-wide scaling vector $\boldsymbol{\beta} = (\beta_{li})_{1 \leq l \leq L, 1 \leq i \leq n_l}$, the achievable rate for the channel in (7) with *i.i.d.* Gaussian input is ([3], [4], [6]):

$$I(P_s, \boldsymbol{\beta}) = (1/2) \log \big(1 + SNR_t\big), \tag{8}$$

where SNR_t, the signal-to-noise ratio at the destination t is:

$$SNR_t = \frac{h_s^2 P_s / \sigma^2}{1 + \sum_{l=1}^{L} \sum_{j=1}^{n_l} h_{lj}^2} \tag{9}$$

For a given network-wide scaling vector $\boldsymbol{\beta}$, the achievable rate of information transfer is given by $I(P_s, \boldsymbol{\beta})$. Therefore the maximum information-rate $I_{ANC}(P_s)$ achievable in a given layered network with *i.i.d.* Gaussian input is defined as the maximum of $I(P_s, \boldsymbol{\beta})$ over all feasible $\boldsymbol{\beta}$, subject to per relay transmit power constraint (3). In other words:

$$I_{ANC}(P_s) \stackrel{def}{=} \max_{\boldsymbol{\beta}: 0 \leq \beta_{li}^2 \leq \beta_{li,max}^2} I(P_s, \boldsymbol{\beta}) \tag{10}$$

It should be noted that $\beta_{li,max}^2$ (the maximum value of the scaling factor for i^{th} node in the l^{th} layer) depends on the scaling factors for the nodes in the previous $l-1$ layers.

Given the monotonicity of the $\log(\cdot)$ function, we have

$$\boldsymbol{\beta}_{opt} = \underset{\boldsymbol{\beta}: 0 \leq \beta_{li}^2 \leq \beta_{li,max}^2}{\text{argmax}} I(P_s, \boldsymbol{\beta}) = \underset{\boldsymbol{\beta}: 0 \leq \beta_{li}^2 \leq \beta_{li,max}^2}{\text{argmax}} SNR_t \tag{11}$$

Therefore in the rest of the paper, we concern ourselves mostly with maximizing the received SNRs.

III. Analyzing the Optimal Performance of Analog Network Coding in General Layered Networks

The problem (11) is a hard optimization problem. In terms of *Geometric Programming* [9], [10], SNR_t is a ratio of posynomials that is a nonlinear (neither convex nor concave) function of $\boldsymbol{\beta}$, in general. It is well-known that maximizing such ratios of posynomials is an intractable problem with no efficient and global solution methods [9, Page 85]. However, globally optimal solutions of such problems can be approximated using heuristic methods based on *signomial programming* condensation that solves a sequence of geometric programs, as in [9, Section 3.3]. Such heuristics though useful

[1]Modified channel gains for even a possibly exponential number of paths as in (5) and (6) can be efficiently computed using the line-graphs [8], and there are only a polynomial number of them in polynomial sized graph.

in providing *good* numerical approximations to the optimal achievable SNR_t, do not provide non-trivial characterization of the optimal SNR_t (or $\boldsymbol{\beta}_{opt}$ that achieves it) in terms of various system parameters. We argue that such characterization however, is highly desired not only for the accurate analysis of ANC performance in general layered networks, but also for various reasons of significant practical consequences, [7].

Towards this goal, in [3], [4] the performance of analog network coding is analyzed under assumptions A and B discussed earlier about per node scaling factor and received SNR at each node, respectively. In the following, we provide an example to establish that the Assumption A, in general, leads to suboptimal ANC rates.

Example 1: Let us consider the well-studied diamond network, [3]. It is defined as $G = (V, E)$ with $V = \{s, t, 1, 2\}$ and $E = \{(s, 1), (s, 2), (1, t), (2, t)\}$. Let $h_{s1} = t^m, h_{s2} = t^p, h_{1t} = t^n, h_{2t} = t^q$ for some $t, t > 0$. The problem of maximum rate achievable with analog network coding for this network can be formulated as (using (11)), [7]:

$$\underset{0\leq\boldsymbol{\beta}^2\leq\boldsymbol{\beta}^2_{max}}{\operatorname{argmax}} \frac{(\beta_1 t^{m+n} + \beta_2 t^{p+q})^2}{1 + \beta_1^2 t^{2n} + \beta_2^2 t^{2q}}, \tag{12}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2)$ and $\boldsymbol{\beta}_{max} = (\beta_{1,max}, \beta_{2,max})$ with $\beta^2_{1,max} = \frac{P_1}{t^{2m}P+\sigma}, \beta^2_{2,max} = \frac{P_2}{t^{2p}P+\sigma}$.

Equating the partial derivatives of the objective function with respect to β_1 and β_2 to zero, we get the following three conditions for global extrema:

$$\beta_1 = -t^{p+q-m-n}\beta_2 \tag{13}$$

$$\beta_1 = t^{m-n-p-q}\beta_2^{-1} + t^{m+q-n-p}\beta_2 \tag{14}$$

$$\beta_2 = t^{p-q-m-n}\beta_1^{-1} + t^{p+n-m-q}\beta_1 \tag{15}$$

It can be proved that (13) leads to the global minimum of the objective function. Further, we can also prove that all choices of the parameters $(t, m, n, p, q, P_s, P_1, P_2)$ that result in one of the constraints $\beta_1^2 < \beta^2_{1,max}$ and $\beta_2^2 < \beta^2_{2,max}$ being satisfied lead to scenarios where global optimum solutions are achieved when the transmit power of one relay node is less than the corresponding maximum, thus contradicting condition (1). For example $(t = 0.1, m = n = q = 0, p = 1, P_s = P_1 = P_2 = 10)$ leads to the optimal solution $(\beta_1 = 0.995, \beta_2 = 0.225)$ whereas $(\beta_{1,max} = 0.995, \beta_{2,max} = 7.07)$, as we show in [6].

Next, we introduce our result that allows us to characterize the optimal performance of analog network coding in general layered networks without the Assumption B or its limited generalization in [4]. This result also provides some key insights into the nature of $\boldsymbol{\beta}_{opt}$ in terms of system parameters.

IV. Computing $\boldsymbol{\beta}_{opt}$ layer-by-layer

In this section we prove that in the end-to-end rate optimal network-wide scaling vector $\boldsymbol{\beta}_{opt}$, the component scaling factors corresponding to the relay nodes in the layer $l, 1 \leq l \leq L$, maximize the sum-rate of the nodes in the layer $l + 1$. However before discussing this result formally, we motivate it by computing the maximum rate of information transfer over a linear amplify-and-forward relay network.

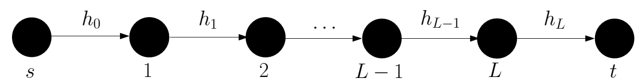

Fig. 1. A linear amplify-and-forward network of $L + 2$ layers. The source s in layer 0, the destination t in layer $L + 1$, and L relays between them.

A. Linear AF Networks

We consider a linear amplify-and-forward network of L relay nodes between the source s and the destination t, as shown in the Figure 1.

Let a feasible scaling vector $\boldsymbol{\beta} = (\beta_1, \dots, \beta_L)$ be such that the output of each relay node satisfies the corresponding transmit power constraint (2). Then the maximum scaling factor for the $l^{\text{th}}, 1 \leq l \leq L$, relay is (from (3)):

$$\beta^2_{l,max} = \frac{P_l}{P_s(h_0 \prod_{i=1}^{l-1} \beta_i h_i)^2 + \sigma^2(1 + \sum_{j=1}^{l-1}(\prod_{i=j}^{l-1} \beta_i h_i)^2)} \tag{16}$$

In a linear AF network, both the source signal and the noise introduced at each intermediate relay node can reach the destination along only one path. Therefore using (5), (6), (7), and (9), for a given scaling vector $\boldsymbol{\beta}$, the received SNR at the destination t or any relay node l can be written as

$$SNR_i = \frac{P_s}{\sigma^2} \frac{(h_0 \prod_{i=1}^{i-1} \beta_i h_i)^2}{1 + \sum_{i=1}^{i-1}(\prod_{j=i}^{i-1} \beta_j h_j)^2}, 1 \leq i \leq L + 1 \tag{17}$$

Lemma 1: The value of β_{L-1} that maximizes SNR_L also maximizes SNR_t.

Proof: The proof involves three steps.

Step 1: Compute the partial derivative of SNR_t with respect to β_L: $\frac{\partial SNR_t}{\partial \beta_L} = 2\frac{P_s h_0^2}{\sigma^2} \frac{(\prod_{i=1}^{L-1} \beta_i h_i)^2 \beta_L h_L^2}{(1+\sum_{i=1}^{L}(\prod_{j=i}^{L} \beta_j h_j)^2)^2}$ This implies that for a given $(\beta_1, \dots, \beta_{L-1})$, SNR_t increases with β_L. However, as the maximum value that β_L can take is $\beta_{L,max}$, so SNR_t attains it maximum value at $\beta_{L,max}$.

Step 2: Using (16) we can express SNR_t only in terms of $(\beta_1, \dots, \beta_{L-1})$ as $SNR_t(\beta_1, \dots, \beta_{L-1})$ given below as

$$\frac{\frac{P_s h_0^2 P_L h_L^2}{\sigma^2}}{P_s h_0^2 + \frac{\sigma^2 + P_L h_L^2}{(\prod_{i=1}^{L-1} \beta_i h_i)^2}(1 + \sum_{i=1}^{L-1}(\prod_{j=i}^{L-1} \beta_j h_j)^2)}$$

Step 3: Compute $\frac{\partial SNR_t(\beta_1, \dots, \beta_{L-1})}{\partial \beta_{L-1}}$, the partial derivative of $SNR_t(\beta_1, \dots, \beta_{L-1})$ with respect to β_{L-1} as

$$\frac{\frac{P_s h_0^2 P_L h_L^2}{\beta_{L-1}}(1 + \frac{P_L h_L^2}{\sigma^2})}{\left[P_s h_0^2 + \frac{\sigma^2 + P_L h_L^2}{(\prod_{i=1}^{L-1} \beta_i h_i)^2}(1 + \sum_{i=1}^{L-1}(\prod_{j=i}^{L-1} \beta_j h_j)^2)\right]^2} \tag{18}$$

Further, from (17) the partial derivative of SNR_L with respect to β_{L-1} evaluates to

$$\frac{\partial SNR_L}{\partial \beta_{L-1}} = 2\frac{P_s h_0^2}{\sigma^2} \frac{(\prod_{i=1}^{L-2} \beta_i h_i)^2 \beta_{L-1} h_{L-1}^2}{(1 + \sum_{i=1}^{L-1}(\prod_{j=i}^{L-1} \beta_j h_j)^2)^2} \tag{19}$$

It follows from (18) and (19) that $SNR_t(\beta_1, \dots, \beta_{L-1})$ and SNR_L are increasing functions of β_{L-1}. Therefore both attain their respective maximum at $\beta_{L-1,max}$, the maximum value of β_{L-1}. In other words, the value of β_{L-1} that maximizes SNR_L also maximizes SNR_t. ■

Following the same sequence of steps as in the proof of above lemma with SNR_t and SNR_L replaced by SNR_L and SNR_{L-1}, respectively, we can also prove that the same value of β_{L-2} (specifically $\beta_{L-2,max}$) maximizes both, SNR_L and SNR_{L-1}. This along with Lemma 1 that allows us to express both, SNR_L and SNR_t as functions of $(\beta_1, \ldots, \beta_{L-2})$, proves that the same value of $\beta_{L-2,max}$ maximizes SNR_{L-1}, SNR_L and SNR_t. Furthermore carrying out this reasoning recursively allows us to express $SNR_i, 2 \leq i \leq L+1$, only in terms of β_1 and to prove that the same value of β_1 (specifically $\beta_{1,max}$) maximizes all of them. We summarize this in the following proposition.

Proposition 1: The scaling vector $\boldsymbol{\beta}_{opt} = (\beta_1^{opt}, \ldots, \beta_L^{opt})$ that solves $\max_{\boldsymbol{\beta}} SNR_t$ can be computed recursively as

$$\beta_i^{opt} = \operatorname*{argmax}_{\beta_i^2 \leq \beta_{i,max}^2} SNR_{i+1}(\beta_1^{opt}, \ldots, \beta_{i-1}^{opt}, \beta_i), 1 \leq i \leq L+1$$

Corollary 1: For a linear AF network with $P_s = P_1 = \ldots = P_L = P$ and $h_0 = h_1 = \ldots = h_L = h$, the maximum achievable information rate $R = \mathcal{O}(1/L)$.

Proof: Using Proposition 1, we can show that

$$(\beta_i^{opt})^2 = \beta_{i,max}^2 = \beta^2 = P/(h^2P + \sigma^2), 1 \leq i \leq L$$

Therefore from (17), we have

$$SNR_t = \left(\frac{h^2P}{\sigma^2}\right)^2 \frac{1-(\beta h)^2}{1-(\beta h)^{2L+2}} (\beta h)^{2L}$$

This implies that the rate $R = \frac{1}{2}\log(1 + SNR_t)$ varies asymptotically with L as $R \leq \frac{1}{2L} \frac{(h^2P/\sigma^2)^2}{1+h^2P/\sigma^2}$. ∎

B. General Layered Networks

We now discuss our result for general layered networks in the general SNR regime.

Lemma 2: Consider a layered relay network of $L+2$ layers, with the source s in layer 0, the destination t in layer $L+1$, and L layers of relay nodes between them. The l^{th} layer contains n_l nodes, $n_0 = n_{L+1} = 1$. The network-wide scaling vector $\boldsymbol{\beta}_{opt} = (\boldsymbol{\beta}_1^{opt}, \ldots, \boldsymbol{\beta}_L^{opt})$ that solves (11) for this network, can be computed recursively for $1 \leq l \leq L$ as

$$\boldsymbol{\beta}_l^{opt} = \operatorname*{argmax}_{\boldsymbol{\beta}_l^2 \leq \boldsymbol{\beta}_{l,max}^2} \prod_{i=1}^{n_{l+1}} (1 + SNR_{l+1,i}(\boldsymbol{\beta}_1^{opt}, \ldots, \boldsymbol{\beta}_{l-1}^{opt}, \boldsymbol{\beta}_l)),$$

where $\boldsymbol{\beta}_l^{opt}$ is the subvector of optimal scaling factors for the nodes in the l^{th} layer, $\boldsymbol{\beta}_l^{opt} = (\beta_{l1}^{opt}, \ldots, \beta_{ln_l}^{opt})$ and constraints $\boldsymbol{\beta}_l^2 \leq \boldsymbol{\beta}_{l,max}^2$ are component-wise $\beta_{li}^2 \leq \beta_{li,max}^2$.

Remark: Lemma 2, in other words, states that the subvector of the optimal scaling vector $\boldsymbol{\beta}_{opt}$, corresponding to the scaling factors of the nodes in the l^{th} layer is the one that maximizes the product $\prod_{i=1}^{n_{l+1}}(1+SNR_{l+1,i})$ of factors $(1+SNR_i), 1 \leq i \leq n_l$, of the n_l nodes in the next $l+1^{\text{st}}$ layer. Now observe that $\log\prod_{i=1}^{n_{l+1}}(1 + SNR_{l+1,i})$ corresponds to $\sum_{i=1}^{n_{l+1}} R_{l+1,i}$, the sum of the information rates to the nodes in the $l+1^{\text{st}}$ layer. Therefore an interpretation of the Lemma 2 is: if starting with the first layer, the scaling factors for the nodes in each successive layer are chosen such that the sum-rate of the nodes in the next layer is maximized, then such a choice also leads to the globally optimal solution of the problem (11).

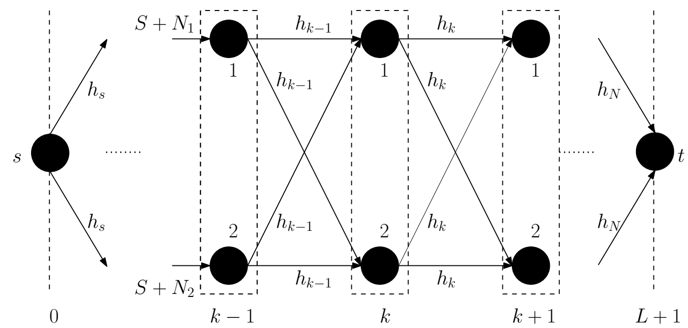

Fig. 2. An ECGAL network of $L+2$ layers, with the source s in layer 0, the destination t in layer $L+1$, and L layers consisting of two relay nodes each between them. The channel gains along all links between two adjacent layers are equal.

Proof: For the ease of presentation, we discuss the proof for a class of layered networks where channel gains along all links between the nodes in two adjacent layers are equal, as in Figure 2. We call such layered networks as *"Equal Channel Gains between Adjacent Layers (ECGAL)"* networks. For the sake of brevity, we omit the details of the proof and refer an interested reader to [7].

Consider the ECGAL network shown in Figure 2. We assume that all nodes have the same transmit power constraint $EX^2 \leq P$. Consider three adjacent layers $k-1, k$, and $k+1$.

Claim: The scaling factors for the nodes in layer $k-1$ that maximize $\prod_{i=1}^{2}(1 + SNR_{k,i})$ also maximize $\prod_{i=1}^{2}(1 + SNR_{k+1,i})$ and vice-versa.

Proof: Let the source signal components[2] of the input at the two nodes in the layer $k-1$ be denoted as S, with $var(S) = S^2$. Let the noise components at the two nodes be denoted as N_1 and N_2, respectively, with $var(N_1) = var(N_2) = N^2$.

The SNRs at the nodes in layers k and $k+1$ are given as:

$$SNR_{k,1} = SNR_{k,2} = \frac{\alpha^2}{\gamma^2}$$

$$SNR_{k+1,1} = SNR_{k+1,2} = \frac{\alpha^2 h_k^2(\beta_3+\beta_4)^2}{\sigma^2 + \gamma^2 h_k^2(\beta_3^2+\beta_4^2)},$$

with $\alpha^2 = S^2 h_{k-1}^2(\beta_1+\beta_2)^2$ and $\gamma^2 = \sigma^2 + h_{k-1}^2 N^2(\beta_1^2+\beta_2^2)$.

Define for $j = k, k+1$

$$SNR_j = \prod_{i \in \{1,2\}} (1 + SNR_{j,i}) = (1 + SNR_{j,1})^2$$

First let us consider the problem

$$\max_{\beta_{k,i}^2 \leq \beta_{k,max}^2} SNR_{k+1}, \tag{20}$$

where $\beta_{k,max}^2 = \frac{P}{\alpha^2+\gamma^2}, i \in \{1,2\}$. It can be proved that $(\beta_{k,max}^2, \beta_{k,max}^2)$ is the only solution of (20).

Substituting the above solution of (20) in the expression for SNR_{k+1} above, allows us to express it in terms of $\beta_{k-1,1}$ and

[2]Given the symmetry of the ECGAL network, source signals at the input of the nodes in every layer are identical.

$\beta_{k-1,2}$. Now consider solutions of the following two problems.

$$\max_{\beta^2_{k-1,i} \leq \beta^2_{k-1,max}} SNR_{k+1}, \quad (21)$$

$$\max_{\beta^2_{k-1,i} \leq \beta^2_{k-1,max}} SNR_{k}, , \quad (22)$$

where $\beta^2_{k-1,max} = \frac{P}{S^2+N^2}, i \in \{1, 2\}$. It can be proved that $(\beta^2_{k-1,max}, \beta^2_{k-1,max})$ is the only solution of both the problems (21) and (22). Thus proving our claim. □

Carrying out the above procedure in the proof of our claim recursively for all $k, 1 \leq k \leq L$ layers, proves the theorem for the ECGAL networks we consider here. ■

V. Illustration

In the following, we provide an example that shows that by using Lemma 2 the maximum rate achievable with analog network coding can be computed even when arbitrarily many layers are not in the high-SNR regime, as required by [3], [4].

Example 2: Consider the ECGAL network of Figure 2 with L layers of relay nodes between the source and the destination and N nodes in each layer. We assume that the channels gains along all links are equal and are denoted as h. For this network, the maximum rate achievable with analog network coding is the solution of the following problem

$$SNR_{t,opt} = \max_{\beta^2 \leq \beta^2_{max}} \frac{h^2 P}{\sigma^2} \frac{(Nh)^{2L} \prod_{i=1}^{L} \beta_i^2}{1 + \sum_{i=1}^{L} (Nh)^{L-i+1} \prod_{j=i}^{L} \beta_i^2} \quad (23)$$

where $\boldsymbol{\beta} = \{\beta_{ij}, 1 \leq i \leq L, 1 \leq j \leq N\}$. From the symmetry of the network, it follows that $\beta^2_{ij,max} = \beta^2_{i,max}$, where

$$\beta^2_{i,max} = P \frac{(h^2P + \sigma^2) \odot (Nh^2P + \sigma^2)^{i-2}}{(h^2P + \sigma^2) \odot (Nh^2P + \sigma^2)^{i-1}}$$

with the operation $\odot$ defined as: $(x + y) \odot (Nx + y)^n = N^{2n}x^{n+1} + \sum_{i=0}^{n-1}(\binom{n}{i} + \binom{n}{i+1})x^{n-i}y^{i+1} + y^{n+1}$.

Using Lemma 2 allows us to prove that the optimal solution of (23) is achieved when $\beta^2_{ij} = \beta^2_{i,max}$. In the following, we discuss this optimal solution in two scenarios.

Case 1: Let $\frac{Nh^2P}{\sigma^2} < 1$, then we have for large L:

$$SNR_{t,max} = \frac{x}{\sigma^{2L}} \frac{x^L(1-x)}{1-x^{L+1}}, x = \frac{Nh^2P}{\sigma^2}$$

In this case, the received signal power at l^{th} layer varies as $SNR \sim (\frac{Nh^2P}{\sigma^2})^L$. Therefore for any fixed δ as in [3], [4], an arbitrarily large number of layers may violate the condition $\min_{k \in l} P_{R,k} \geq 1/\delta, l = 1, \ldots, L$ as L grows. Thus the approaches in [3], [4] cannot be used to reasonably characterize the maximum achievable rates in such networks, as we computed above.

Case 2: Consider the scenario when $\frac{Nh^2P}{\sigma^2} \gg 1$. In this case, we have

$$SNR_{t,max} = N \frac{\frac{x}{x+1}}{1 - (\frac{x}{x+1})^L}, x = \frac{Nh^2P}{\sigma^2}$$

Therefore, the maximum rate achievable with analog network coding in this scenario is $R_{ANC} = \frac{1}{2}\log(1 + SNR_{t,max})$ which approaches the MAC cut-set bound $C = \frac{1}{2}\log(1+Nx)$ [6], within a constant gap as $x \to \infty$, as shown in the Figure 3.

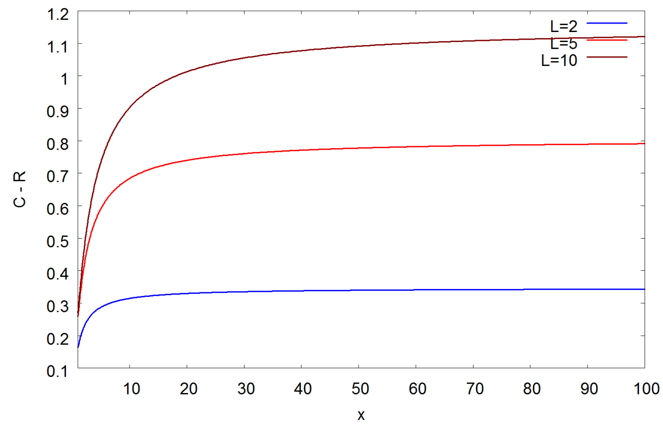

Fig. 3. For the ECGAL networks with equal channel gains: the gap between MAC cut-set bound C and analog network coding rate R as the parameter $x = \frac{Nh^2P}{\sigma^2}$ increases. The number of nodes in each layer is $N = 5$. We observe that for a given number of layers in the network the gap approaches a constant value. As the number of layers in the network increases, the corresponding gap also increases.

VI. Conclusion and Future Work

We consider the problem of maximum rate achievable with analog network coding in the general layered networks. Previously, this problem was addressed assuming that the nodes in all but at most one layer in the network are in the high-SNR regime, and each node forwards the received signal at the upper bound of its transmit power constraint. We provide a key result that allows us to exactly compute the maximum ANC rate in a wider class of layered network without these assumptions. Further, our result significantly reduces the computational complexity of solving this problem for general layered networks. We illustrate the significance of our result by computing the maximum ANC rate for one particular relay network in a scenario that cannot be addressed using existing approaches. In the future, we plan to extend this work to non-layered networks, and to construct the optimal distributed relay schemes.

References

[1] S. -Y. R. Li, R. W. Yeung, and N. Cai, "Linear network coding," *IEEE Trans. Inform. Theory,* vol. IT-49, February 2003.

[2] S. Katti, S. Gollakotta, and D. Katabi, "Embracing wireless interference: analog network coding," *Proc. ACM SIGCOMM,* Kyoto, Japan, August, 2007.

[3] I. Marić, A. Goldsmith, and M. Médard, "Analog network coding in the high-SNR regime," *Proc. IEEE WiNC 2010,* Boston, MA, June 2010.

[4] B. Liu and N. Cai, "Analog network coding in the generalized high-SNR regime," *Proc. IEEE ISIT 2011,* St. Petersburg, Russia, July-August 2011.

[5] K. S. Gomadam and S. A. Jafar, "Optimal relay functionality for SNR maximization in memoryless relay networks," *IEEE JSAC,* vol. 25, February 2007.

[6] S. Agnihotri, S. Jaggi, and M. Chen, "Amplify-and-Forward in Wireless Relay Networks," *Proc. IEEE ITW 2011,* Paraty, Brazil, October 2011.

[7] S. Agnihotri, S. Jaggi, and M. Chen, "Analog Network Coding in General SNR Regime," *arXiv:??.*

[8] R. Koetter and M. Médard, "An algebraic approach to network coding," *IEEE/ACM Trans. Netw.,* vol. 11, October 2003.

[9] M. Chiang, *Geometric Programming for Communication Systems.* now Publishers Inc., Boston, 2005.

[10] S. Boyd, S. -J. Kim, L. Vandenberghe, and A. Hassibi, "A tutorial on geometric programming," *Optim. Eng.,* vol. 8, April 2007.