Revenue Maximization for Communication Networks with Usage-Based Pricing

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***Abstract*—We study the optimal usage-based pricing problem in a resource-bounded network with one profit-maximizing service provider and multiple groups of surplus-maximizing users. We first analytically derive the optimal pricing mechanism that the service provider maximizes the service provider's revenue under complete network information. Then we consider the incomplete information case, and propose two incentive compatible pricing schemes that achieve different complexity and performance tradeoff. Finally, by properly combining the two pricing schemes, we can show that it is possible to maintain a very small revenue loss (e.g., 0.5% in a two-group case) without knowing detailed information of each user in the network.**

I. Introduction

Since Kelly's seminal work [1] [2], pricing has been widely adopted to study various resource allocation problems in communication networks. The past decade has witnessed its development from the original application in Transmission Control Protocol (TCP) to the more general framework of Network Utility Maximization (NUM) theory [3]. In various NUM formulation, prices mainly serve as indicators of the dual variables of the network optimization problem, and coordinate the system resource allocation to maximize the social welfare (e.g., the summation of all users' utilities).

In reality, however, a selfish network service provider (SP) is more likely to set prices to maximize its own revenue. This is the focus of this paper and we try to answer the following two questions:

1) Given complete information of the network users, how should the SP set the revenue maximizing prices?
2) If only partial (e.g., statistical) information is known, how should the SP set prices to achieve (close to) optimal revenue?

To make the study concrete, we study the revenue maximization problem of a single resource-limited SP facing multiple groups of users. Each user determines its optimal resource demand to maximize the surplus (i.e., the difference between its utility and payment). The SP chooses the prices to maximize its revenue, subject to limited amount of total resource.

The key results and contributions of this paper are as follows:

- *Optimal pricing scheme under complete information*: we show that the optimal pricing scheme involves price differentiation of various user groups with a water-filling solution structure. Although this result is not surprising from a microeconomic point of view, several interesting properties derived from the optimal pricing scheme provide important insights for designing (sub-)optimal pricing schemes under incomplete information.
- *Incentive-compatible nonlinear pricing with incomplete information*: when the SP knows the statistical network information but not each individual user's utility function, we design a nonlinear (piecewise linear) pricing scheme that encourages each individual user to choose the same quantity and price as in the optimal pricing scheme under complete information. We provide the necessary and sufficient conditions under which such scheme achieves the optimal revenue.
- *Linear pricing scheme with incomplete information*: we further simplify the pricing mechanism and derive the linear pricing, which shares a similar water-filling structure as the optimal pricing differentiation scheme, but can be implemented much more easily.
- *Bounded revenue loss in the incomplete information case*: by using a combination of the nonlinear and linear pricing schemes, we show that the revenue loss is negligible (e.g., less than 0.5%) in the two-group case. This shows that with a properly designed pricing mechanism, we can achieve the close-to-optimal revenue without knowing the detailed information of each user in the network.

Our paper is the first paper that analytically solved the usage-based pricing revenue maximization problem for a resource-constrained monopolistic service provider, under both complete and incomplete information scenarios. Our system model is somewhat similar to that considered in [4], which focused on the bandwidth allocation in a single link network. Reference [4] focused on the congestion-limited case where the performance penalty goes to infinity when the total resource usage is close to the link's limited capacity. As a result, the resource constraint is never reached in [4]. In contrast, we consider the case where the total resource can be fully allocated to all users, which leads to a water-filling solution structures in all three pricing mechanisms. The other difference is that we focus on presenting the solution for a network with an arbitrary finite number of users, whereas [4] focused on presenting nice insights in the asymptotic case of many users.

Recently [5] proposed the novel concept of *price of simplicity*, and showed a single entry fee only leads to small revenue loss compared to the price differentiation strategy in many interesting cases. While whether network should be charged based on fixed entry fee or usage-based prices has long been

TABLE I
A COMPARISON OF THE PRICE OF SIMPLICITY

Price of simplicity	Flat entry pricing	Usage-based linear pricing
Reference	Recent Work [5]	This Paper
How to solve the SP's problem?	search the marginal user	search the threshold of a water-filling solution
Complexity	$\mathcal{O}(M)$, where M is the total number of users	$\mathcal{O}(I)$, where I is the total number of user groups
Admission control	Needed	Not needed
Resource allocation	forced by SP	based on users' local optimizations
Loss performance	bounded (<13%)	might be high in some case

a subject of much controversy, our result of linear pricing can be viewed as a parallel result of "price of simplicity" for the usage-based pricing. Table I shows the comparison between our paper and [5].

Other work of pricing and revenue management in communication network includes [6]–[8]. Much of this work focused on the study of the interaction between different service providers embodied in the pricing strategies, rather than to design the pricing mechanism as considered in this paper.

In the next section, we introduce the system model and analyze the optimal pricing scheme. Based on this result, in Section III, we study two incentive compatible pricing schemes, and provide both analysis and numerical results for the revenue loss between them and the optimal pricing schemes. The paper is concluded in Section IV.

II. SYSTEM MODEL AND OPTIMAL PRICING UNDER COMPLETE INFORMATION

We consider a network with a total amount of S limited resource[1]. The resource is allocated by a single service provider (SP) to a set $\mathcal{I} = \{1, \dots, I\}$ of user groups. Each group $i \in \mathcal{I}$ has N_i homogeneous users.[2] Each user in group i has the same utility function:

$$u_i(s_i) = \theta_i \ln(1 + s_i), \tag{1}$$

where s_i is the allocated resource to that user and θ_i represents the willingness to pay of group i. Without loss of generality, we assume that $\theta_1 > \theta_2 > \cdots > \theta_I$. Here we choose the logarithmic function so that we can express quantities of interest in closed forms. We feel that the insights we gain by using this simplification justify it. Generalization of the utility functions will be discussed in Section II-C.

We consider two types of information structures:

- *Complete information*: the SP knows each user's utility function.
- *Incomplete information*: the SP knows the total number of groups I, the number of users in each group N_is, and the utility function of each group u_is. It does not know which user belongs to which group. Such statistical information can be obtained through long term observations of a stationary user population.

The interaction between the SP and users can be characterized as a two-stage process shown in Fig. 1. The details depend on the information structure of the SP. In this section we will consider the case of complete information. In this case,

[1]The resource can be in the form of rate, bandwidth, power, time slot, etc.

[2]A special case is N_i=1 for each group, i.e., all users in the network are different.

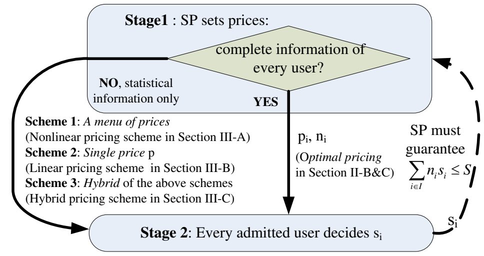

Fig. 1. A two-stage system model

the profit-maximizing SP decides the price p_i and admitted number of users n_i for every group i in the first stage. In the second stage, an admitted user in group i chooses the resource quantity s_i to maximize the difference between its utility and payment, i.e., the surplus. We will analyze the process using backward induction.

A. User's Surplus Maximization Problem in Stage 2

If a user in group i is admitted into the network in stage 1, then it solves the following surplus maximization problem under a fixed price p_i,

$$\max_{s_i \geq 0} \; u_i(s_i) - p_i s_i,$$

which leads to the following unique solution

$$s_i(p_i) = \left(\frac{\theta_i}{p_i} - 1\right)^+,$$

where $(x)^+ = \max(x, 0)$.

B. SP's Pricing and Admission Control Problem in Stage 1

The SP's problem in the first stage can be formulated as Problem P_0, with an objective of maximizing its revenue subject to the limited total resource. The key idea is to perform price differentiation , i.e., charge each group i with a different price p_i.

$$P_0: \quad \max \quad \sum_{i \in \mathcal{I}} n_i p_i s_i$$

$$\text{s.t.} \quad s_i = \left(\frac{\theta_i}{p_i} - 1\right)^+, \;\; i \in \mathcal{I}, \tag{2}$$

$$n_i \in \{0, \dots, N_i\} \;, \;\; i \in \mathcal{I}, \tag{3}$$

$$\sum_{i \in \mathcal{I}} n_i s_i \leq S \tag{4}$$

$$\text{variables}: \;\; \boldsymbol{p} \geq \boldsymbol{0}, \; \boldsymbol{n}$$

where $\boldsymbol{p}$ and $\boldsymbol{n}$ are vector forms of p_is and n_is. We use the bold font denote the vector in the sequel. Constraint (3) represents the admission control, and constraint (4) represents the total limited resource in the network.

Problem P_0 is hard to solve directly, since it is is a non-convex optimization with a non-convex objective (summation of products of n_i and p_i), the coupled constraint (4), and integer variables $\boldsymbol{n}$. However, it is possible to transform it into an equivalent convex formulation. According to (2), there is no need for the SP to set p_i higher then θ_i for users in group i; otherwise users in group i will demand zero resource and generate zero revenue. This means that we can rewrite constraint (2) as

$$p_i = \frac{\theta_i}{s_i + 1} \text{ and } s_i \geq 0, i \in \mathcal{I}. \tag{5}$$

Plug (5) into Problem P_0, we have the following equivalent problem representation:

$$\begin{aligned} P_1: \quad & \max \quad \sum_{i\in\mathcal{I}} n_i \frac{\theta_i s_i}{s_i+1} \\ & \text{s.t.} \quad n_i \in \{0,\ldots,N_i\}, \quad i\in\mathcal{I} \\ & \qquad \sum_{i\in\mathcal{I}} n_i s_i \le S \\ & \text{variables}: \quad \boldsymbol{s} \ge \boldsymbol{0},\ \boldsymbol{n} \end{aligned}$$

Note that for a given $\boldsymbol{n}$, the objective function in Problem P_1 is strictly concave in $\boldsymbol{s}$. We can solve Problem P_1 by sequentially solving two sub-problems:

1) the *resource allocation problem*: for a fixed $\boldsymbol{n}$, maximize the objective over $\boldsymbol{s}$.
2) the *admission control problem*: plug the solution of the resource allocation problem, then maximize the objective over $\boldsymbol{n}$.

The solutions are summarized in the following lemmas (detailed proof in Appendix):

Lemma 1: Given admission control decision $\boldsymbol{n}$, the unique optimal solution of the resource allocation problem in Problem P_1 is

$$s_i^* = \left(\sqrt{\frac{\theta_i}{\lambda^*}} - 1\right)^+, \quad i\in\mathcal{I}, \tag{6}$$

where λ^* is the unique solution of the weighted water-filling problem $\sum_{i\in\mathcal{I}} n_i \left(\sqrt{\frac{\theta_i}{\lambda}} - 1\right)^+ = S$.

The water-filling problem in general has no closed-form solution. However, we can efficiently determine the water-level by exploiting the special structure of the problem. Note that since $\theta_1 > \theta_2 > \cdots > \theta_I$, the water level $\frac{1}{\sqrt{\lambda^*}}$ must satisfy the following condition: $\sum_{i=1}^K n_i(\sqrt{\frac{\theta_i}{\lambda^*}} - 1) = S$, for a threshold value K where $\frac{\theta_K}{\lambda^*} > 1$ and $\frac{\theta_{K+1}}{\lambda^*} \le 1$. This leads to the following simple algorithm to search λ^*:

Algorithm 1: denote k as the iteration index.

- *Initiation:* $k = I$;
- *Step 1:* calculate $\lambda(k) = \left(\frac{\sum_{i=1}^k n_i\sqrt{\theta_i}}{S+\sum_{i=1}^k n_i}\right)^2$;
- *Step 2:* check whether $\theta_k > \lambda(k)$
  - If NO, $k \leftarrow k-1$, GO TO *Step 1*;
  - If YES, $K \leftarrow k$ and $\lambda^* \leftarrow \lambda(k)$.
- *Termination:* RETURN K and λ^*.

Since $\theta_1 > \lambda(1)$ holds, the algorithm always stops and returns the exact value of λ^*. The complexity is $\mathcal{O}(I)$, i.e., linear in the total number of user groups (not the total number of users). Similar algorithms for solving different types of water-filling problems have been proposed in [9], [10].

After finding K and λ^*, we can update the solution of the resource allocation problem in (6) as

$$s_i^* = \begin{cases} \sqrt{\frac{\theta_i}{\lambda^*}} - 1 & i = 1,2,\ldots,K; \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The threshold structure implies that the resource is allocated to the higher willingness to pay users with priority under the optimal pricing scheme. By the Algorithm 1, we have $\lambda^* = (\frac{\sum_{i=1}^K n_i\sqrt{\theta_i}}{S+\sum_{i=1}^K n_i})^2$. Thus the threshold condition $\frac{\theta_K}{\lambda^*} > 1$ can be equivalently written as $\sqrt{\theta_K} > \frac{\sum_{i=1}^{K-1} n_i\sqrt{\theta_i}}{S+\sum_{i=1}^{K-1} n_i}$. Since $\theta_1 > \theta_2 > \cdots > \theta_{K-1} > \theta_K$, we can derive the following necessary condition at the threshold,

$$\sqrt{\theta_K} > \frac{\sum_{i=1}^{K-1} n_i}{S+\sum_{i=1}^{K-1} n_i}\sqrt{\theta_{K-1}}. \tag{8}$$

Now let us consider a case where n_i (for a group index i less than K) increases to infinity, in which case formula (8) will definitely be violated. This means that K is no longer the threshold group index in this limiting case, Therefore, we see that whether a group K user can receive resource or not is determined by the numbers of the users in the groups with higher willingness to pay. If there are too many high willingness to pay users, then the low willingness to pay users will not be allocated any resource.

Now let us consider the admission problem of determining the optimal $\boldsymbol{n}^*$, which can be solved based on Lemma 1. We can prove that the objective of this integer variable maximization problem is strictly increasing in n_i, $\forall\ i \le K$ and in independent of n_i, $\forall\ i > K$ (detailed proof in Appendix). Therefore, we have the following lemma[3]:

Lemma 2: It is optimal to admit all users in the network in Problem P_1, i.e., $n_i^* = N_i$, $i\in\mathcal{I}$.

By Lemma 1 and Lemma 2, we can obtain the optimal solution of the SP's Problem P_0:

Theorem 1: An optimal solution of Problem P_0 is

- *Admission Control:*

$$n_i^* = N_i, \quad i\in\mathcal{I}. \tag{9}$$

- *Optimal pricing:*

$$p_i^* = \begin{cases} \sqrt{\theta_i\lambda^*}, & i = 1,2,\ldots,K, \\ \theta_i, & \text{otherwise.} \end{cases} \tag{10}$$

where λ^* and K can be obtained by Algorithm 1 by $n_i = N_i$, $i\in\mathcal{I}$.

Theorem 1 provides the right economic intuition: SP maximizes its revenue by doing price differentiation, by charging a higher price to users with a higher willingness to pay. It is easy to check $p_i > p_j$ for any $i < j$. Moreover, prices for groups larger than K are so high that the users in these groups will receive zero resource.

There are several other interesting properties of the optimal pricing scheme. For example, the resource constraint (4) is always tight, which means the resource allocation is Pareto-optimal under the optimal pricing. Moreover, (9) shows that the SP should not perform any admission control, since the users have elastic data applications and the revenue is increasing in the number of the admitted users. Finally, the optimal prices can be viewed as congestion indicators of the scarce network resource. From (7) and (10), it is easy to see that the p_i^*, ($\forall\ i\in\mathcal{I}$) increases and resource s_i^*, ($\forall\ i \le K$) decreases as any the number of users n_i^*, ($\forall\ i \le K$) increases.

[3]We note that there are several optimal solutions to the admission control problem of Problem P_1, since for a group $i > K$ we can set n_i^* to be any integer no larger than N_i. But any user from such a group will request zero resource, and thus it is enough to consider the case where all users from all groups are admitted.

C. Optimal Pricing under General Utility Functions

It is possible to solve the optimal pricing problem P_0 under general utility functions, with the following two classes explained in more details in the technical report Appendix:

- $u_i(s) = \theta_i u(s)$, where $u(s)$ is a strictly increasing and concave function (not necessarily logarithmic) with some mild technical conditions. This includes the well studied α-fair utility function [11].
- $u_i(s) = \theta_i \log(1 + h_i s_i)$, which is motivated by the Shannon capacity in wireless communication networks where s_i is the allocated downlink power and h_i is the normalized SNR per unit power. Notice that if each group contains only one user, then this models a general wireless network where every user has arbitrary willingness to pay and channel condition.

III. Incentive Compatible Pricing Schemes under Incomplete Information

The optimal price differentiation pricing mechanism in Theorem 1 can only be implemented if the SP knows the complete network information, i.e., knowing which group each user belongs to and charging accordingly. In the case of incomplete information, however, a user in group i can pretend that it is a user of group $q(> i)$ in an attempt to be charged with a lower price. Thus it is important to design incentive compatible pricing schemes such that a user does not have the incentive to lie about its true type (i.e., which group it belongs to). Here we will design two pricing mechanisms (nonlinear and linear) that are incentive compatible and lead to zero or small revenue loss when properly used.

A. Incentive-compatible Nonlinear Pricing

We first design a nonlinear pricing scheme that mimics the optimal price differentiation in Theorem 1. Since there are only K groups of users receiving non-zero resource allocations in Theorem 1, we propose a nonlinear price menu with K prices, $p_1^* > p_2^* > \cdots > p_K^*$. These prices are exactly the same optimal prices that the SP would charge for the K groups as in Theorem 1. Note that for the $K+1, \ldots, I$ groups, all the prices in the menu are too high for them, then they will still demand zero resource. Since we know that a user with a higher θ_i will demand more resource under the optimal pricing, we divide the quantity into K intervals by $K-1$ thresholds, $s_{th}^1 > s_{th}^2 > \cdots > s_{th}^{K-1}$. The complete incentive compatible nonlinear pricing mechanism is specified as follows:

$$p_N(s) = \begin{cases} p_1^* & \text{when } s > s_{th}^1 \\ p_2^* & \text{when } s_{th}^1 \geq s > s_{th}^2 \\ \vdots & \\ p_K^* & \text{when } s_{th}^{K-1} \geq s > 0. \end{cases} \quad (11)$$

Under this new pricing scheme, the SP publishes the quantity-based price menu (11), and allowes the users to freely choose their quantities. As showed in Fig. 2, the price is a piecewise linear function in quantity s. In contrast to the traditional volume discount, the nonlinear pricing mechanism implies a larger unit price for a larger quantity. This is motived by the optimal pricing in Theorem 1, and the quantity is used as an index of the user's willingness to pay to make the users *self-differentiated*. The key challenge here is to properly set the quantity thresholds so that users are perfectly segmented through self-differentiation. This is, however, not always possible. Next we derive the necessary and sufficient conditions to guarantee the perfect segmentation.

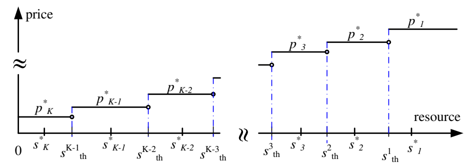

Fig. 2. Nonlinear pricing mechanism: where the prices satisfy $p_1^* > p_2^* \cdots > p_K^*$, and are set as the same as the optimal pricing. s_{th}^{q-1}s $(q = 2, \ldots, K)$ are the thresholds for this piecewise-linear pricing scheme. To mimic the same resource allocation as under the optimal pricing mechanism, one necessary condition is $s_{th}^{q-1} \geq s_q^*$ for all q, where s_q^* is the corresponding resource allocation under the optimal pricing mechanism.

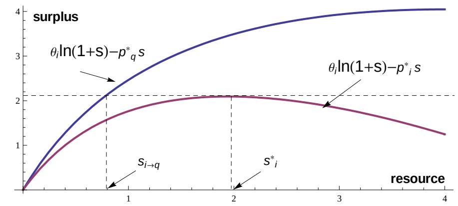

Fig. 3. When the threshold $s_{th}^{q-1} < s_{i\to q}$, the group i user can not obtain $U(s_i^*, p_i^*)$ if it chooses the lower price p_q at a quantity less than s_{th}^{q-1}. Therefore it will automatically choose the high price p_i^* to maximize its surplus.

Let us first study the self-selection problem between two groups: group i and group q with $i < q$. Later on we will generalize the results to multiple groups. Define $U_i(s; p_q^*)$ as the surplus of a group i user when it is charged with price p_q^*:

$$U_i(s; p_q^*) = \theta_i \ln(1 + s) - p_q^* s.$$

If we still use s_i^* to denote the optimal resource allocation under the optimal pricing as in Theorem 1, then we have

$$s_i^* = \arg\max_{s_i \geq 0} U_i(s_i; p_i^*).$$

We use $s_{i\to q}$ (with $i < q$) to denote the quantity that satisfies the following relationships,

$$\begin{cases} U_i(s_{i\to q}; p_q^*) = U_i(s_i^*; p_i^*) \\ s_{i\to q} < s_i^*. \end{cases} \quad (12)$$

In other words, when a group i user is charged with a lower price p_q^* and demands resource quantity at $s_{i\to q}$, it achieves the maximum surplus as under the optimal pricing p_i^*, as showed in Fig. 3.

To maintain the group i users' incentive to choose the higher price p_i^* instead of p_q^*, we must have $s_{th}^{q-1} \leq s_{i\to q}$, which means a group i user can not obtain $U(s_i^*, p_i^*)$ if it chooses a quantity less than s_{th}^{q-1}. In other words, it will automatically choose the higher (and the desirable) price p_i^* to maximize its surplus.

On the other hand, we must have $s_{th}^{q-1} \geq s_q^*$ in order to maintain the optimal resource allocation and allow a group q user to choose the right quantity-price combination (illustrated in Fig. 2).

Therefore, it is clear that the *necessary and sufficient* condition that the nonlinear pricing mechanism under incomplete information achieves the same maximum revenue of the optimal pricing under complete information is

$$s_q^* \leq s_{i\to q},\ \forall\, i<q, \forall\, q \in \{2,\ldots,K\}. \quad (13)$$

Solving these inequalities, we can obtain the following theorem (detailed proof in Appendix).

Theorem 2: For any fixed total resource S and user populations $\{N_1,\ldots,N_K\}$, there exists unique thresholds of $\{t_1,\ldots,t_{K-1}\}$, such that the nonlinear pricing achieves the same maximum revenue as in the complete information case if $\sqrt{\frac{\theta_q}{\theta_{q+1}}} \geq t_q$ for $q=1,\ldots,K-1$. Moreover, t_q is the unique solution of the equation $t^2\ln t-(t^2-1)+\frac{t\sum_{k=1}^{q}N_k+N_{q+1}}{S+\sum_{k=1}^{K}N_k}(t-1)=0$ over the domain $t>1$.

We want to mention that the conditions in Theorem 2 is necessary and sufficient for the case of $K=2$ active groups[4]. For $K>2$, it is difficult to precisely solve (13) and thus Theorem 2 is sufficient but not necessary.

The following result immediately follows Theorem 2.

Corollary 1: The t_qs in Theorem 2 satisfy $t_q < t_{root}$ for $q=1,\ldots,K-1$, where $t_{root}\approx 2.21846$ is the larger root of equation $t^2\ln t-(t^2-1)=0$.

When the conditions in Theorem 2 are not satisfied, there may be loss in terms of the SP's revenue. Since it is difficult to explicitly solve the parameterized transcend equation (12), analytically characterization of the loss is not yet possible.

To tackle this difficulty, we introduce the linear pricing mechanism next. This can be viewed as a degenerated case of the nonlinear pricing. There is no issue of truthful revealing of group types, since all groups are charged with the same unit price. It is clear that in general the linear pricing scheme will suffer a positive revenue loss by giving up price differentiation.

B. Linear Pricing Scheme

Let us consider the following problem where the SP maximizes its profit by setting the same unit price p to all groups:

$$\begin{aligned} P_{sp}: \quad & \max \quad p\sum_{i\in\mathcal{I}} n_i s_i \\ & \text{s.t.} \quad s_i=\left(\frac{\theta_i}{p}-1\right)^+,\ i\in\mathcal{I} \quad (14)\\ & n_i\in\{0,\ldots,N_i\}\ ,\ i\in\mathcal{I} \\ & \sum_{i\in\mathcal{I}} n_i s_i \leq S \quad (15)\\ \text{variables}: \quad & p\geq 0,\ \boldsymbol{n} \end{aligned}$$

After transformation, we find this problem is equivalent to the weighted water-filling problem $\sum_{i\in\mathcal{I}} N_i(\frac{\theta_i}{p}-1)^+=S$. Thus we can obtain the following optimal solution that shares a similar structure as the optimal pricing with complete information.

Theorem 3: An optimal solution of the linear pricing problem P_{sp} is:

- *Admission control:* $n_i=N_i,\ i\in\mathcal{I}$.
- *Optimal single price:* $p=p(K_0)=\frac{\sum_{i=1}^{K_0}N_i\theta_i}{S+\sum_{i=1}^{K_0}N_i}$, where $K_0 \leq I$ is a positive integer satisfying the threshold structure $\frac{\theta_{K_0}}{p(K_0)}>1$, and $\frac{\theta_{K_0+1}}{p(K_0)}\leq 1$.
- The corresponding *resource allocation* for the single price mechanism: $s_i=\begin{cases}\frac{\theta_i}{p}-1 & i=1,2,\ldots,K_0,\\ 0 & \text{otherwise.}\end{cases}$

Compared with the optimal pricing scheme in Theorem 1, the linear pricing scheme also gives a higher priority to users with a higher willingness to pay, but typically with a different cutting-off group threshold K_0.

Based on the results in Theorems 3 and 1, we are able to calculate the revenue loss in the linear pricing case. For the general case with arbitrary number of groups, however, the number of parameters is too large to make the comparison insightful. Next we will focus the comparison in the two-group case and show the linear pricing may lead to a large revenue loss. Then we show that a combination of the nonlinear pricing and the linear pricing (i.e., scheme 3 in Fig. 1) can significantly reduce the revenue loss.

C. Revenue Loss Analysis in a Two-Group Case

In a two-group case, the revenue under the linear pricing scheme in Theorem 3 is

$$R_{sp}=\begin{cases}\frac{S(N_1\theta_1+N_2\theta_2)}{N_1+N_2+S} & 1\leq t<\sqrt{\frac{S+N_1}{N_1}},\\ \frac{SN_1\theta_1}{N_1+S} & t\geq\sqrt{\frac{S+N_1}{N_1}},\end{cases}$$

where $t=\sqrt{\frac{\theta_1}{\theta_2}}>1$. The optimal revenue achieved with full information in Theorem 1 is

$$R_{opt}=\begin{cases}\frac{S(N_1\theta_1+N_2\theta_2)+N_1N_2(\sqrt{\theta_1}-\sqrt{\theta_2})^2}{N_1+N_2+S} & 1\leq t<\frac{S+N_1}{N_1},\\ \frac{SN_1\theta_1}{N_1+S} & t\geq\frac{S+N_1}{N_1}.\end{cases}$$

Let us define the revenue loss ratio

$$L=\frac{R_{opt}-R_{sp}}{R_{opt}},\ \ t>1.$$

Let $N=N_1+N_2$ be the total number of the users, $\alpha=\frac{N_1}{N}$ be the percentage of group 1 users, and $k=\frac{S}{N}$ denotes the level of normalized available resource. Thus the loss ratio $L(t,\alpha,k)=$

$$\begin{cases}\frac{\alpha(1-\alpha)(t-1)^2}{k(\alpha t^2+1-\alpha)+\alpha(1-\alpha)(t-1)^2} & 1<t<\sqrt{\frac{k+\alpha}{\alpha}},\\ \frac{(1-\alpha)(\alpha(t-1)-k)^2}{(\alpha+k)((\alpha t^2+1-\alpha)k+\alpha(1-\alpha)(t-1)^2)} & \sqrt{\frac{k+\alpha}{\alpha}}\leq t<\frac{k+\alpha}{\alpha}.\end{cases}$$

It is clear that $L=0$ when $t\geq\frac{k+\alpha}{\alpha}$. The following properties of L are interesting:

- For parameter t: L is an increasing function of t in $(1,\sqrt{\frac{k+\alpha}{\alpha}})$ and a decreasing function in $[\sqrt{\frac{k+\alpha}{\alpha}},\frac{k+\alpha}{\alpha})$. The maximum is obtained at $t_{L-max}=\sqrt{\frac{k+\alpha}{\alpha}}$, which is a critical point that the resource allocated to the group 2 user just becomes zero under the linear pricing. One example is showed in fig.4.
- For parameter α: We define

$$\begin{aligned} L(\alpha,k) &= \max_t L(t,\alpha,k) \\ &= \frac{(1-\alpha)(\sqrt{k+\alpha}-\sqrt{\alpha})^2}{k(k+1)+(1-\alpha)(\sqrt{k+\alpha}-\sqrt{\alpha})^2}. \end{aligned}$$

  For a fixed k, $L(\alpha,k)$ monotonically decreases in α. This is essentially due to the fact that both optimal and linear pricing schemes allocate resource to the high willingness to pay users with priority due to the limited total resource.

[4]There might be other groups who are not allocated positive resource under the optimal pricing.

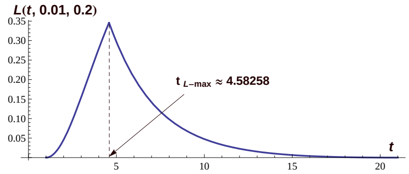

Fig. 4. The loss ratio L increases in t in $(1, \sqrt{\frac{k+\alpha}{\alpha}})$, and decreases in $[\sqrt{\frac{k+\alpha}{\alpha}}, \frac{k+\alpha}{\alpha})$. The maximum value is achieved at $t_{L-max} = \sqrt{\frac{k+\alpha}{\alpha}}$.

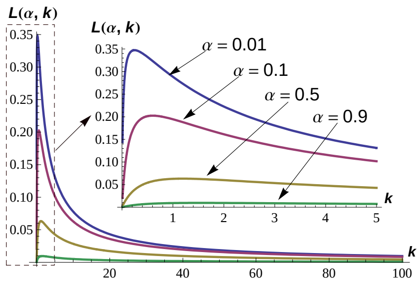

Fig. 5. For a fixed k, L monotonically increases in α. For a fixed α, L first increases in k, and then decreases in k.

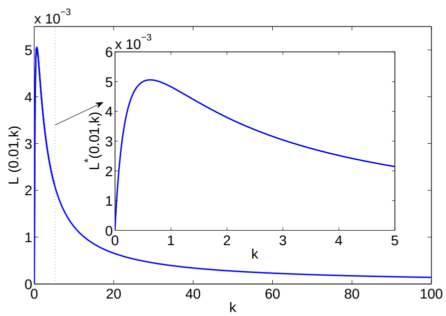

Fig. 6. Illustration of the maximum possible loss ratio $L^*(\alpha, k)$ with $\alpha = 0.01$ of the hybrid pricing scheme. $L^*(0.01, k)$ is bounded by 0.5%, which is much less than the $L(0.01, k)$ in Fig. 5.

When the ratio of the high willingness to pay users (α) increases, the differences between two pricing strategies and their corresponding resource allocations diminish. As shown in Fig. 5, when α is large (≥ 0.5), the revenue loss are very small ($L(\alpha, k) < 7\%$).

- For parameter k: For fixed α, $L(\alpha, k)$ is not a monotonic function in k. As showed in Fig. 5, $L(\alpha, k)$ is small when k is very small or very large. Small k means that the resource is very limited and almost exclusively allocated to group 1's users in both pricing schemes. When k is very large, the resource is abundant and the prices and resource allocation in two strategies also become close.

From the above analysis, we find the following hybrid pricing scheme is promising to serve as an implementable pricing mechanism for the SP in the incomplete information case.

$$p(t) = \begin{cases} \text{linear pricing} & \text{when } t < t_1, \\ \text{nonlinear pricing} & \text{when } t \geq t_1, \end{cases}$$

where t_1 is the threshold obtained by Theorem 2. When $t \geq t_1$, the nonlinear pricing can maintain zero revenue loss. When $t < t_1$, a big loss happens in the linear pricing scheme only if α is small and t is near t_{L-max}. However, in the hybrid pricing mechanism, when α is small, t_1 is always much smaller than t_{L-max}. Therefore we do not observe big loss.

To illustrate the benefit of the hybrid pricing scheme, we numerically compute the maximum possible loss ratio $L^*(\alpha, k) = L(\min(t_1, t_{L-max}), \alpha, k)$ to illustrate the above intuition. As showed in Fig. 6, the loss ratio $L^*(0.01, k)$ for the hybrid pricing is bounded under 0.5%. On the other hand, the maximum loss ratio $L(\alpha, k)$ for single price can reach near 35% (see Fig. 5) with the same parameters.

IV. Conclusion

In this paper, we first study the maximum revenue that can be achieved by a monopolistic service provider under complete network information. Then we propose two pricing schemes with incomplete information, and show that by properly combining the two schemes we will have very small revenue loss in a two-group case while maintaining the incentive compatibility. The ongoing work involves analytically characterizing the efficiency loss in the general network with arbitrary number of user groups.

Appendix

A. Proofs of Lemma 1 and Lemma 2

Proof: In Problem P_1, since for a given $\boldsymbol{n}$, the objective is strictly concave over $\boldsymbol{s}$, thus to solve this problem is equivalent to solve the following two problems sequentially.
the *Resource allocation problem*:

$$\begin{aligned} P_1^1: \quad & \max_{s_i \geq 0} \quad \sum_{i \in \mathcal{I}} n_i \frac{\theta_i s_i}{s_i + 1} \\ & \text{s.t.} \quad \sum_{i \in \mathcal{I}} n_i s_i \leq S \\ & \text{variables} \quad \boldsymbol{s} \end{aligned} \tag{16}$$

Denote the solution of P_1^1 as $s_i^*(\boldsymbol{n})$. Then, we solve the optimization problem with the integer variables $\boldsymbol{n}$.
the *Admission control problem*:

$$\begin{aligned} P_1^2: \quad & \max \quad \sum_{i \in \mathcal{I}} n_i \frac{\theta_i s_i^*(\boldsymbol{n})}{s_i^*(\boldsymbol{n}) + 1} \\ & \text{s.t.} \quad n_i \in \{0, \ldots, N_i\}, \quad i \in \mathcal{I} \\ & \text{variables} \quad \boldsymbol{n} \end{aligned} \tag{17}$$

Problem P_1^1 is a convex optimization problem. Note that the resource constraint (16) must be satisfied with equality, since the objective is strictly increasing function with s_i, and considering if not, it means that there is unused resource and then the SP can make more revenue by increasing the resource given to any group.

Then we use the Lagrange multiplier technique to solve Problem P_1^1. Let λ be the Lagrange multiplier corresponding to the resource constraint. Then, the Lagrangian for this problem is given by

$$L(\boldsymbol{s}, \lambda) = \sum_{i \in \mathcal{I}} n_i \frac{\theta_i s_i}{s_i + 1} - \lambda(\sum_{i \in \mathcal{I}} n_i s_i - S).$$

With respect to s_i, $i = 1, 2, \ldots I$, we obtain the first-order necessary condition:

$$\frac{\partial L(\boldsymbol{s}, \lambda)}{\partial s_i} = \frac{\theta_i n_i}{(s_i + 1)^2} - \lambda n_i = 0.$$

Since $s_i \geq 0$, it follows

$$s_i^*(\lambda) = (\sqrt{\frac{\theta_i}{\lambda}} - 1)^+. \tag{18}$$

Using the fact that equality holds in (16), we can get λ by solving

$$\sum_{i\in\mathcal{I}} n_i \max(\sqrt{\frac{\theta_i}{\lambda}}-1,0)=S.$$

This weighted water-filling problem can be solved by the aforementioned Algorithm 1:

$$\sqrt{\lambda^*}=\frac{\sum_{i=i}^{K} n_i\sqrt{\theta_i}}{S+\sum_{i=1}^{K} n_i}. \quad (19)$$

where K is some threshold computed by the algorithm, thus the solution of Problem P_1^1 can be further simplified as

$$s_i^*=\begin{cases}\sqrt{\frac{\theta_i}{\lambda^*}}-1 & i=1,2,\ldots,K;\\ 0 & \text{otherwise.}\end{cases}$$

Plug this result to problem P_1^2, then the objective can be equivalently transformed as $\sum_{i=1}^{K} n_i s_i\sqrt{\theta_i}\sqrt{\lambda^*}$. Since in (16), we have $\sum_{i=1}^{K} n_i s_i=S$, and it is not hard to check from (19) that $\sqrt{\lambda^*}$ is an increasing function in n_i, $i=1,\ldots,K$, thus the objective is also an increasing function in $n_i,\ i\in\{1\ldots,K\}$. We note that there are several optimal solutions to the admission control problem of Problem P_1^2, since the objective is nothing to do with $n_i,\ i\in\{K+1,\ldots,I\}$, thus for a group $i>K$, we can set n_i^* to be any integer no larger than N_i. But any user from such a group will request zero resource, and thus it is enough to consider the case where all users from all groups are admitted. Therefore, one of the optimal solutions of the problem P_1^2 is $n_i^*=N_i$ for all $i\in\mathcal{I}$. ∎

B. Optimal Pricing under General Utility Functions

We solve the optimal pricing problem P_0 under general utility functions with the following two classes. Since the solving technique and procedure are almost the same as mentioned in Section II-B, we omit some of the details of the derivation here.

1) $u_i(s)=\theta_i u(s)$, where $u(s)$ is a strictly increasing and concave function (not necessarily logarithmic).

We need the following technical condition to guarantee that Problem P_0 can be transformed into a convex optimization formulation:

$$\frac{\partial^3 u(s)}{\partial s^3}s+2\frac{\partial^2 u(s)}{\partial s^2}<0. \quad (20)$$

Under this condition, we can first solve the the resource allocation problem as follows:

Lemma 3: Given $\boldsymbol{n}$, the unique optimal solution of the resource allocation problem in Problem P_1 is

$$s_i^* = \begin{cases} g^{-1}(\frac{\lambda^*}{\theta_i}) & i=1,2,\ldots,K_{th}\\ 0 & \text{otherwise}\end{cases} \quad (21)$$

where $g(s_i)=u''(s_i)s_i+u'(s_i)$, g^{-1} is its inverse function[5], λ^* is the unique solution of the weighted water-filling problem

$$\sum_{i\in\mathcal{I}} n_i(g^{-1}(\frac{\lambda}{\theta_i}))^+=S, \quad (22)$$

and K_{th} is the corresponding threshold, which can be obtained in a similar way as Algorithm 1.

Lemma 4: It is optimal to admit all users in the network in Problem P_1, i.e., $n_i^*=N_i,\ i\in\mathcal{I}$.

Proof: When plugging (21) in (22), we have

$$\sum_{i=1}^{K_{th}} n_i s_i^*=\sum_{i=1}^{K_{th}} n_i g^{-1}(\frac{\lambda}{\theta_i})=S. \quad (23)$$

If we increase any n_i, $i=1,2,\ldots,K_{th}$, then $g^{-1}(\frac{\lambda}{\theta_i})$, $i=1,2,\ldots,K_{th}$ must decrease to hold the equality (23), (i.e., λ increases). Therefore we can see that s_i^* is a strictly decreasing function in n_i, $i=1,2,\ldots,K_{th}$. On the other hand, $p_i=\theta_i u'(s_i^*)$, is a strict decreasing function of s_i^*, therefore, p_i is a strictly increasing function of n_i, $i=1,2,\ldots,K_{th}$.

The objective of the admission control problem P_1^2, the total revenue, can be written as

$$\sum_{i=1}^{K_{th}} p_i n_i s_i^*=\bar{p}\sum_{i=1}^{K_{th}} n_i s_i^*=\bar{p}S,$$

where $\bar{p}$ is the average unit price $\bar{p}=\sum_{i=1}^{K_{th}}\gamma_i p_i$ with the non-negative weight γ_is satisfying $\sum_{i=1}^{K_{th}}\gamma_i=1$. When any n_i increases, it may lead to the changes the weights. However, p_i, $i=1,2,\ldots,K_{th}$ strictly increases, then $\bar{p}$ strictly increases. Therefore, it is optimal to admit all the users in group i, $i=1,2,\ldots,K_{th}$.

Since the objective is nothing to do with $n_i,\ i\in\{K_{th}+1,\ldots,I\}$, thus for a group $i>K_{th}$, we can set n_i^* to be any integer no larger than N_i. But any user from such a group will request zero resource, and thus it is enough to consider the case where all users from all groups are admitted. Therefore, one of the optimal solutions of the problem P_1^2 is $n_i^*=N_i$ for all $i\in\mathcal{I}$. ∎

Thus, by the above two lemma, we can solve the Problem P_0 as follows.

Theorem 4: An optimal solution of Problem P_0 is

- *Admission Control:*

$$n_i^*=N_i,\ \ i\in\mathcal{I}.$$

- *Optimal pricing:*

$$p_i^*=\begin{cases}\theta_i u'(g^{-1}(\frac{\lambda^*}{\theta_i})) & i=1,2,\ldots,K_{th},\\ \theta_i u'(0) & \text{otherwise;}\end{cases} \quad (24)$$

where λ^* and K_{th} are obtained by Lemma 3 when $n_i=N_i$, $i\in\mathcal{I}$.

2) $u_{ij}(s_{ij})=\theta_i\log(1+h_{ij}s_{ij})$, which is motivated by the Shannon capacity in wireless communication networks where s_{ij} is the allocated downlink power of one of the N_i users in group i and h_{ij} is the normalized SNR per unit power. When we apply this utility function into Problem P_0, we will get the following results:

Lemma 5: One of the optimal admission control strategy is to admit all users in the network, i.e., $n_i^*=N_i,\ i\in\mathcal{I}$.

This result is easy to see. Since we consider different resource quantity variable s_{ij} for each user, setting the admission control rule as $n_i^*=N_i$ will not affect the

[5]Since (20) guarantees that $g(s_i)$ is a monotonic function, there exists its inverse function g^{-1}. However, it may not have a explicit form solution. In spite of this, If all the parameters are given, we can first numerically compute g^{-1}, and then obtain the solution.

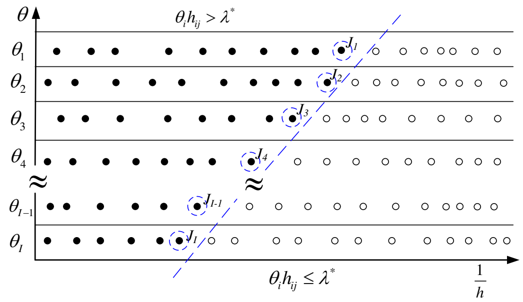

Fig. 7. Threshold structure for the solution of the resource allocation with Shannon-type utility: in the $\theta - \frac{1}{h}$ plane, the black dots denote the users with corresponding θ_i and h_{ij} getting positive resource, while write dots represente the users getting zero resource under the optimal pricing scheme. The black dots and the white dots are separated by the line $\theta_i h_{ij} = \lambda^*$.

optimal solution. In other words, in this situation, the admission control can be equivalently realized by setting zero resource quantity to the users that should not be admitted.

Based on Lemma 5, we can solve the resource allocation problem, and summarize the result in the following lemma.

Lemma 6: The unique optimal solution of the resource allocation problem in Problem P_1 is

$$s_{ij}^* = \begin{cases} \frac{1}{h_{ij}}(\sqrt{\frac{\theta_i h_{ij}}{\lambda^*}} - 1) & j = 1, \ldots, J_i \;\; i \in \mathcal{I}, \\ 0 & j = J_i + 1, \ldots, N_i \; i \in \mathcal{I}. \end{cases}$$

where λ^* is the unique solution of the water-filling problem $\sum_{i=1}^{I}\sum_{j=1}^{N_i} \frac{1}{h_{ij}}(\sqrt{\frac{\theta_i h_{ij}}{\lambda}} - 1)^+ = S$, and J_is ($i \in \mathcal{I}$) are corresponding thresholds.

Proof: To solve two-dimension (i, j) water-filling problem

$$\sum_{i=1}^{I}\sum_{j=1}^{N_i} \frac{1}{h_{ij}}(\sqrt{\frac{\theta_i h_{ij}}{\lambda}} - 1)^+ = S, \tag{25}$$

we first construct a one-to-one mapping from the two-dimension parameters $\{\theta_i h_{ij}\}$, $j \in \{1, 2, \ldots, N_i\}$, $i \in \mathcal{I}$ to one dimension sequence $\{\theta_k h_k\}$, $k \in \{1, 2, \ldots, K\}$ where $K = \sum_{i \in \mathcal{I}} N_i$. The sequence $\{\theta_k h_k\}$ is a non-increasing sequence, i.e., $\theta_1 h_1 \geq \theta_2 h_2 \cdots \geq \theta_K h_K$. We denote this mapping as $\mathcal{F}$. By this new notation, (25) can be equivalently written as $\sum_{k=1}^{K} \frac{1}{h_k}(\sqrt{\frac{\theta_k h_k}{\lambda}} - 1)^+ = S$. This one-dimension water-filling problem can be solved by the similar way as Algorithm 1, thus we obtain the threshold K_0 and λ^*, such that for all $k < K_0$, $s_k^* = \frac{1}{h_k}(\sqrt{\frac{\theta_i h_k}{\lambda^*}} - 1)$ and for all $k \geq K_0$, $s_k^* = 0$.

After obtaining the solution of resource allocation problem, we do the reverse mapping $\mathcal{F}^{-1}$, and we will find the threshold for every group, that is, for group i, we have a J_i, for all $j \leq J_i$, $s_{ij}^* = \frac{1}{h_{ij}}(\sqrt{\frac{\theta_i h_{ij}}{\lambda^*}} - 1)$, and for all $j > J_i$, $s_{ij}^* = 0$. ■

Fig. 7 illustrates one interesting property of it. When we plot this resource allocation result in the $\theta - \frac{1}{h}$ plane, we will see that the threshold structure is a line that separates the plane into two parts, with users in one part getting non-zero resource, and the remaining users in the other part getting zero resource. Further, the part with users getting non-zero resource has a larger product of willingness to pay and the channel gain. Thus we find that the priority of resource allocation in the wireless case is not only determined by the users' willingness to pay, but their channel conditions as well.

By Lemma 5 and Lemma 6, we can obtain the optimal solution of Problem P_0:

Theorem 5: An optimal solution of Problem P_0 is

- *Admission Control:*

$$n_i^* = N_i, \;\; i \in \mathcal{I}.$$

- *Optimal pricing:*

$$p_{ij}^* = \begin{cases} \sqrt{\theta_i \lambda^* h_{ij}} & j = 1, 2, \ldots, J_i; \;\; \forall i \\ \sqrt{\theta_i h_{ij}} & j = J_i + 1, \ldots, N_i; \;\; \forall i \end{cases}$$

where λ^* and J_is are obtained by Lemma 6.

C. Proof of Theorem 2 and Corollary 1

Proof: Since $U_i(s, p_q)$ is a strictly increasing function in the interval $[0, s_i^*]$, that the inequality (13) holds, if and only if the following inequality holds:

$$U_i(s_q^*, p_q) \leq U_i(s_{i \to q}, p_q), \; \forall i < q. \tag{26}$$

After simplifying, (26) is equivalent to

$$\theta_i \ln \sqrt{\frac{\theta_i}{\theta_q}} - (\theta_i - \theta_q) + \sqrt{\lambda^*}(\sqrt{\theta_i} - \sqrt{\theta_q}) \geq 0, \; \forall i < q.$$

where λ^* is given by (19) with $n_i = N_i$, $i \in \mathcal{I}$.

Let both sides of the inequality be divided by θ_q, and denote $t_{iq} = \sqrt{\frac{\theta_i}{\theta_q}}$, we have:

$$t_{iq}^2 \ln t_{iq} - (t_{iq}^2 - 1) + \frac{\sum_{k=1}^{K} N_k t_{kq}}{\sum_{k=1}^{K} N_k + S}(t_{iq} - 1) \geq 0, \; \forall i < q. \tag{27}$$

Since $t_{1q} > \cdots > t_{Kq}$, (27) can be equivalently simplified as:

$$t_{q-1q}^2 \ln t_{q-1q} - (t_{q-1q}^2 - 1) + \frac{\sum_{k=1}^{K} N_k t_{kq}}{\sum_{k=1}^{K} N_k + S}(t_{q-1q} - 1) \geq 0. \tag{28}$$

For convenience, we abbreviate t_{q-1q} as t_q, $(q = 2, \ldots, K)$ in the sequel. It is easy to see that the following inequality is the necessary and sufficient condition of (28) for $q = 2$, and sufficient condition of (28) for $q > 2$:

$$t_q^2 \ln t_q - (t_q^2 - 1) + \frac{t_q \sum_{k=1}^{q-1} N_k + N_q}{\sum_{k=1}^{K} N_k + S}(t_q - 1) \geq 0. \tag{29}$$

Let $g(t)$ be the left hand side of the inequality (29). It is easy to check that $g(t)$ is a convex function, with $g(1) = 0$, $g(\infty) = \infty$ and $g'(1) < 0$. So there exists a root $t_q > 1$. When $t > t_q$, the inequality (29) holds, thus (26) holds, and the conclusion in Theorem 2 is followed.

Further, it is also easy to check $g(t)$ is increasing over $N_1, \ldots, N_{K-1}$, and $1/S$. Thus put all the parameters to the extreme case, we can find the upper bound for t_0. Then we have $t_q < t_{root}$ for any $N_1, \ldots, N_{q-1}$, and S, where t_{root} is the larger root of $t^2 \ln t - (t^2 - 1) = 0$, numerically, $t_{root} \approx 2.21846$. ■

D. Proof of Theorem 3

Proof: Similar analysis as the optimal pricing problem P_0, we sequentially maximize the objective of Problem P_{sp} over the different variables.

The *resource allocation* problem: given $\boldsymbol{n}$, and canceling out $\boldsymbol{s}$ by (14), it is easy to see, the objective is a decreasing function over p, and by (15), the minimum p is achievable when the equality holds. Thus this problem is equivalent to the following water-filling problem:

$$\sum_{i\in\mathcal{I}} n_i(\frac{\theta_i}{p}-1)^+ = S.$$

We can solve it by the similar way as Algorithm 1, thus we can obtain a threshold K_0 and

$$p^*(\boldsymbol{n}) = p(K_0) = \frac{\sum_{i=1}^{K_0} n_i\theta_i}{S+\sum_{i=1}^{K_0} n_i} \tag{30}$$

satisfying

$$\frac{\theta_{K_0}}{p(K_0)} > 1, \text{ and } \frac{\theta_{K_0+1}}{p(K_0)} \leq 1.$$

By (15), we can get the solution of the resource allocation problem:

$$s_i = \begin{cases} \frac{\theta_i}{p^*(\boldsymbol{n})} - 1 & i = 1,2,\ldots,K_0, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

The *admission control* problem: plug (30) and (31) into the objective and maximize it on the variable $\boldsymbol{n}$. The objective is equivalent to $p^*(\boldsymbol{n})S$. It is not hard to check from (30) that $p^*(\boldsymbol{n})$ is an increasing function of $n_i,\ i\in\{1\ldots,K_0\}$. Since the objective is nothing to do with $n_i,\ i\in\{K_0+1,\ldots,I\}$, thus for a group $i > K_0$, we can set n_i^* to be any integer no larger than N_i. But any user from such a group will request zero resource, and thus it is enough to consider the case where all users from all groups are admitted. Therefore, one of the optimal solutions of the admission control problem is

$$n_i^* = N_i,\ i\in\mathcal{I}. \tag{32}$$

We plug (32) into (30) and (31), thus we obtain the solution the linear pricing problem P_{sp} just as showed in Theorem 3. ■

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