Autonomous Spectrum Balancing (ASB) for Frequency Selective Interference Channels

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***Abstract*— For frequency selective interference channels where interference is treated as noise, distributively attaining the boundary of the rate region is an open problem, and is particularly important for broadband DSL access. This paper develops, analyzes, and simulates a new algorithm for power allocation in frequency selective interference channels called Autonomous Spectrum Balancing (ASB). It utilizes the concept of a "reference line", which mimics a typical victim line in the interference channel. Compared with the state-of-the-art Iterative Watefilling and Optimum Spectrum Balancing methods, the ASB algorithm is completely autonomous, has linear complexity in both the number of users and tones, and gives close to near-optimal performance. Convergence of a version of ASB is proven for any number of users.**

I. Overview

Frequency selective interference channels are frequently encountered in many practical communication systems. In this paper we consider a specific case of the interference channel, in which each receiver is forced to treat interference from other transmitters as noise. We investigate the problem of optimizing the transmit spectra of the different transmitters in a distributed fashion, in order to operate as close to the boundary of the rate region as possible. This problem is non-convex and coupled across users and tones. It is particularly important for digital subscriber line (DSL) systems, where modems have the ability to shape their transmit spectra, and adapt the power spectrum density (PSD) used within each sub-channel or tone, but must treat interference from other modems as noise.

In the *iterative waterfilling* (IW) algorithm, each line maximizes its own data rate by waterfilling over the noise and interference from other lines [1]. IW is completely autonomous algorithm with a linear complexity in the number of users. Unfortunately, although IW can achieve near optimal performance in weak interference channels, it is highly-suboptimal in near-far scenarios, such as mixed central office (CO) / remote terminal (RT) deployments of ADSL and upstream VDSL, because of the greedy nature of the algorithm. The *optimal spectrum balancing* (OSB) algorithm addresses this problem through maximization of a weighted rate-sum that explicitly takes into account the damage done to the other lines within the network when optimizing each line's spectra [2]. Unfortunately OSB has an exponential complexity in the number of users, making it extremely complex when the DSL system contains many lines. Furthermore, the OSB algorithm is not distributed, instead relying on a centralized network management center (NMC) to optimize the PSDs for all modems. This NMC requires knowledge of the crosstalk channels between all lines, something that is often not available in practice. In recent work, *iterative spectrum balancing* (ISB) was proposed, which implements the weighted-rate sum optimization of OSB in an iterative fashion over the users [3], [4]. This leads to a quadratic complexity in the number of users, however ISB still requires centralized operation.

This paper proposes a new algorithm: *autonomous spectrum balancing* (ASB), for spectra optimization in frequency selective interference channels, and particularly for dynamic spectrum management in DSL. ASB is autonomous: it can be applied in a distributed fashion across users with no explicitly information exchange. Furthermore, the algorithm has a linear complexity in both the number of users and tones, and is provably convergent under reasonable conditions on the channel gains that are often satisfied in DSL. The proposed algorithm is also shown to achieve near-optimal performance, operating close to the globally optimal rate region, which could previously only be achieved through the centralized, and highly complexity OSB. The basic idea behind ASB is to leverage the fact that DSL interference channel gains are very slowly time-varying, which enables an effective use of the concept of "reference line" that represents a typical victim within a DSL system. When adapting its PSD, each line attempt to achieve its own target rate whilst minimizing the damage it does to the reference line, thereby achieving a reasonable balance between selfish and socially responsible operation. We prove the convergence of ASB under an arbitrary number of users, for both sequential and parallel updates. Since IW can be recovered as a special case of ASB, our work extends previous work on IW [5], giving a simpler and more general proof of convergence. Table I compares various aspects of different power allocation algorithms, where ASB attains the best tradeoff among distributiveness, complexity, and performance.

II. System Model

Using the notation as in [2], [3], we consider a DSL model of a frequency selective interference channel with a set $\mathcal{N} = \{1, ..., N\}$ modems (i.e., lines, users) and $\mathcal{K} = \{1, ..., K\}$ tones (i.e., frequency carriers). Assuming the standard discrete multi-tone (DMT) modulation is applied, transmission can be modeled independently on each tone as follows:

$$\boldsymbol{y}_k = \boldsymbol{H}_k \boldsymbol{x}_k + \boldsymbol{z}_k.$$

The ASB algorithm was first developed by Raphael Cendrillon while under the supervision of Marc Moonen at the Katholieke Universiteit Leuven, Belgium. Further developments of ASB and convergence proofs were obtained together with Jianwei Huang and Mung Chiang at Princeton University. This work was carried out in the frame of IUAP P5/22, *Dynamical Systems and Control: Computation, Identification and Modelling* and P5/11, *Mobile multimedia communication systems and networks*; the Concerted Research Action GOA-MEFISTO-666, *Mathematical Engineering for Information and Communication Systems Technology*; IWT SOLIDT Project, *Solutions for xDSL Interoperability, Deployment and New Technologies*; FWO Project G.0196.02, *Design of efficient communication techniques for wireless time-dispersive multi-user MIMO systems*. The work was partially sponsored by Alcatel-Bell, by US NSF Grant CNS-0427677 *FAST Copper Project*, and by US NSF CAREER Grant CCF-0448012.

TABLE I
COMPARISON OF POWER ALLOCATION ALGORITHMS

Algorithm	Operation	Complexity	Performance	Reference
IW	Autonomous	$O(KN)$	Sub-optimal	[1]
OSB	Centralized	$O\left(Ke^N\right)$	Optimal	[2]
ISB	Centralized	$O\left(KN^2\right)$	Near optimal	[3], [4]
ASB	Autonomous	$O(KN)$	Near optimal	this paper

The vector $\boldsymbol{x}_k \triangleq \{x_k^n, n \in \mathcal{N}\}$ contains transmitted signals on tone k, where x_k^n is the signal transmitted by line n at tone k. Vectors $\boldsymbol{y}_k$ and $\boldsymbol{z}_k$ have similar structures: $\boldsymbol{y}_k$ is the vector of received signals on tone k, and $\boldsymbol{z}_k$ is the vector of additive noise on tone k and contains thermal noise, alien crosstalk and radio frequency interference. We denote the channel gain from transmitter m to receiver n on tone k as $h_k^{n,m}$. We denote the transmit power spectrum density (PSD) $s_k^n \triangleq \mathcal{E}\left\{|x_k^n|^2\right\}$, the vector containing the PSD of user n on all tones as $\boldsymbol{s}^n \triangleq \{s_k^n, k \in \mathcal{K}\}$.

Assume that each modem treats interference from other modems as noise. When the number of interfering modems is large, the interference can be well approximated by a Gaussian distribution. Under this assumption the achievable bit rate of user n on tone k is

$$b_k^n \triangleq \log\left(1 + \frac{1}{\Gamma}\frac{s_k^n}{\sum_{m\neq n}\alpha_k^{n,m}s_k^m + \sigma_k^n}\right), \qquad (1)$$

where $\alpha_k^{n,m} = |h_k^{n,m}|^2 / |h_k^{n,n}|^2$ is the normalized crosstalk channel gain (with $\alpha_k^{n,n} = 0, \forall k, n$), and σ_k^n is the noise power density normalized by the direct channel gain $|h_k^{n,n}|^2$. Here Γ denotes the SINR-gap to capacity, which is a function of the desired BER, coding gain and noise margin [6]. For notational simplicity, we absorb Γ into the definition of $\alpha_k^{n,m}$ and σ_k^n. The bandwidth of each tone is normalized to 1. The data rate on line n is thus $R^n = \sum_{k\in\mathcal{K}} b_k^n$. Each modem n is typically subject to a total power constraint P^n, due to the limitations on each modem's analog frontend: $\sum_{k\in\mathcal{K}} s_k^n \le P^n$.

III. Spectrum Management Problem Formulation

The spectrum management problem is defined as follows

$$\max_{\{\boldsymbol{s}^n, n\in\mathcal{N}\}} R^1 \quad \text{s.t. } R^n \ge R^{n,\text{target}}, \forall n > 1, \qquad (2)$$
$$\text{s.t. } \sum_{k\in\mathcal{K}} s_k^n \le P^n, \forall n.$$

Here $R^{n,\text{target}}$ denote the target rate of user n. Due to interference between lines, problem (2) is nonconvex. Furthermore, it is highly coupled across lines and tones, making it a very difficult optimization to solve. In particular, any algorithm that globally solves (2) must have knowledge of all crosstalk channels and background noise spectra, forcing it to operate in a centralized fashion. In order to overcome this difficulty, we observe that for optimal solutions of (2) each user adopts a PSD that achieves a fair compromise between maximizing their own data-rate and minimizing the damage they do to other lines within the network. Here the definition of fair depends on the target rate of each individual line.

Based on this insight, we introduce the concept of a "reference line", a virtual line that represents a "typical" victim within the DSL system. Since network operators are typically concerned with maximizing the rate achieved by the worst line within their network, the reference line typically corresponds to the the longest line in the network (e.g. the CO distributed line in a mixed CO/RT scenario, such as that in Section VI), which has the weakest direct channel and receives relatively stronger crosstalk from other users. Then instead of solving (2) , each user tries to maximize the achievable rate on the reference line, subject to its own rate and total power constraints.

Assuming that only user n causes interference to the reference line, we can determine the rate achieved by the reference line as

$$R^{n,ref} \triangleq \sum_{k\in\mathcal{K}} \tilde{b}_k^n \triangleq \sum_{k\in\mathcal{K}} \log\left(1 + \frac{\tilde{s}_k}{\tilde{\alpha}_k^n s_k^n + \tilde{\sigma}_k}\right).$$

The coefficients $\{\tilde{s}_k, \tilde{\sigma}_k, \tilde{\alpha}_k^n, \forall k, n\}$ are parameters of the reference line and can be obtained from long-term field measurements[1]. Since the crosstalk channel can be regarded as time-invariant in the DSL wireline network, the parameters of the reference lines are known to users a priori. Intuitively, the reference line serves a penalty term in each user's optimization problem to avoid purely selfish behavior, and eliminates the need of explicit message passing amongst users.

The optimization of user n can now be stated as

$$\max_{\boldsymbol{s}^n} R^{n,ref} \quad \text{s.t. } R^n \ge R^{n,\text{target}}, \qquad (3)$$
$$\text{s.t. } \sum_{k\in\mathcal{K}} s_k^n \le P^n.$$

By incorporating the target rate constraints implicitly through a weight coefficient (dual variable) w^n, problem (3) can be restated as

$$\max_{\boldsymbol{s}^n} w^n R^n + R^{n,ref} \quad \text{s.t. } \sum_{k\in\mathcal{K}} s_k^n \le P^n. \qquad (4)$$

By adjusting the weight w^n such that user n's target rate constraint becomes tight, problem (4) becomes the dual of problem (3). Since the number of tones is typically large in DSL (e.g, 512 for VDSL), this dual-based approach leads to an asymptotic optimal solution of the original Problem (3) [2].

IV. ASB Algorithms

A. ASB-I

We first introduce the basic version of the ASB algorithm (ASB-I), where each user n adjusts their PSD $\boldsymbol{s}^n$ in order to solve optimization (4). Thue user then updates their weight w^n such that they achieve the respective target rate. For each user, replacing the original optimization Problem (4) with the Lagrange dual problem

$$\max_{\substack{\lambda^n\ge 0 \\ \sum_{k\in\mathcal{K}} s_k^n \le P^n}} \sum_{k\in\mathcal{K}} \max_{s_k^n} J_k^n\left(w^n, \lambda^n, s_k^n, s_k^{-n}\right), \qquad (5)$$

where the object for user n on tone k is

$$J_k^n\left(w^n, \lambda^n, s_k^n, s_k^{-n}\right) = w^n b_k^n + \tilde{b}_k^n - \lambda^n s_k^n, \qquad (6)$$

and b_k^n depends on both user n's PSD s_k^n and all other users' PSDs s_k^{-n} on the same tone k. By introducing the dual variable

[1]In fact, the reference line concept is already used in existing VDSL standards such as T1.424-2004. Good choices for reference lines have been defined based on extensive studies. However, it has not been used for PSD optimization as proposed in the ASB algorithm.

λ^n, we decouple Problem (4) into several smaller subproblems, one for each tone. The optimal PSD that maximizes J_k^n for a fixed w^n and λ^n is

$$s_k^{n,I}\left(w^n,\lambda^n,s_k^{-n}\right)=\arg\max_{s_k^n\in[0,P^n]} J_k^n\left(w^n,\lambda^n,s_k^n,s_k^{-n}\right), \tag{7}$$

which can be found by solving the first order condition, $\partial J_k^n\left(w^n,\lambda^n,s_k^n,s_k^{-n}\right)/\partial s_k^n=0$, which leads to

$$\frac{w^n}{s_k^{n,I}+\sum_{m\neq n}\alpha_k^{n,m}s_k^m+\sigma_k^n} - \frac{\tilde{\alpha}_k^n\tilde{s}_k}{\left(\tilde{s}_k+\tilde{\alpha}_k^n s_k^{n,I}+\tilde{\sigma}_k\right)\left(\tilde{\alpha}_k^n s_k^{n,I}+\tilde{\sigma}_k\right)}-\lambda^n=0. \tag{8}$$

Equation (8) can be simplified into a cubic equation which has three solutions. The optimal PSD can be found by substituting these three solutions back to the objective function $J_k^n\left(w^n,\lambda^n,s_k^n,s_k^{-n},\right)$, as well as checking the boundary solutions $s_k^n=0$ and $s_k^n=P^n$, and choosing the one that yields the largest value of J_k^n.

The user then updates λ^n to enforce the total power constraint, and updates w^n to enforce the target rate constraint. The complete algorithm is given as follows, where ε_λ and ε_w denote the stepsizes for λ^n and w^n, and $[x]^+\triangleq\max\{x,0\}$.

Algorithm 1 ASB-I

```
repeat
    for each user n = 1, ..., N
        repeat
            for each tone k = 1, ..., K, find
                s_k^{n,I} = arg max_{s_k^n >= 0} J_k^n
            λ^n = [λ^n + ε_λ (Σ_k s_k^{n,I} − P^n)]^+ ;
            w^n = [w^n + ε_w (R^{n,target} − Σ_k b_k^n)]^+ ;
        until convergence
    end
until convergence
```

B. ASB-II with Frequency-Selective Waterfilling

We now introduce a variation of the ASB algorithm (ASB-II) that enjoys even lower lower computational complexity and has provable convergence. To obtain the optimal PSD in ASB-I (for fixed λ^n and w^n), we had to solve the roots of a cubic equation on each tone. To reduce the computation complexity and gain more insight into the solution structure, we assume that the reference line operates in the high SINR regime whenever it is active, that is iff $\tilde{s}_k>0$, then $\tilde{s}_k\gg\tilde{\sigma}_k\gg\alpha_k^{n,m}s_k^n$ for any feasible s_k^n, $n\in\mathcal{N}$ and $k\in\mathcal{K}$. This assumption is motivated by our observations of optimal solutions for DSL interference channels. Intuitively, we assume that the received signal power on the reference line is much larger than the reference noise, which is in turn much larger than the interference from user n. Thus on any tone where the reference line is active, $k\in\bar{\mathcal{K}}\triangleq\{k|\tilde{s}_k>0,k\in\mathcal{K}\}$, the achievable rate is

$$\log\left(1+\frac{\tilde{s}_k}{\tilde{\alpha}_k^n s_k^n+\tilde{\sigma}_k}\right)\approx\log\left(\frac{\tilde{s}_k}{\tilde{\sigma}_k}\right)-\frac{\tilde{\alpha}_k^n s_k^n}{\tilde{\sigma}_k}.$$

and user n's objective function on tone k can be approximated by

$$J_k^{n,II,1}\left(w^n,\lambda^n,s_k^n,s_k^{-n}\right)=w^n b_k^n-\frac{\tilde{\alpha}_k^n s_k^n}{\tilde{\sigma}_k}-\lambda^n s_k^n+\log\left(\frac{\tilde{s}_k}{\tilde{\sigma}_k}\right).$$

The corresponding optimal PSD is

$$s_k^{n,II,1}\left(w^n,\lambda^n,s_k^{-n}\right)=\left[\frac{w_n}{\lambda^n+\tilde{\alpha}_k^n/\tilde{\sigma}_k}-\sum_{m\neq n}\alpha_k^{n,m}s_k^m-\sigma_k^n\right]^+. \tag{9}$$

This is a water-filling type of solution and is intuitively satisfying as will now be explained. First of all the PSD for user n should be smaller when the power constraint is tighter (i.e., λ_n is larger), or the crosstalk channel to the reference line $\tilde{\alpha}_k^n$ is higher, or the noise level on the reference line $\tilde{\sigma}_k$ is smaller, or there is more interference plus noise $\sum_{m\neq n}\alpha_k^{n,m}s_k^m+\sigma_k^n$ on the current tone. [2]

On the other hand, on any tone where the reference line is inactive, i.e., $k\in\bar{\mathcal{K}}^C\triangleq\{k|\tilde{s}_k=0,k\in\mathcal{K}\}$, the objective function is

$$J_k^{n,II,2}\left(w^n,\lambda^n,s_k^n,s_k^{-n}\right)=w^n b_k^n-\lambda^n s_k^n,$$

and the corresponding optimal PSD is

$$s_k^{n,II,2}\left(w^n,\lambda^n,s_k^{-n}\right)=\left[\frac{w^n}{\lambda^n}-\sum_{m\neq n}\alpha_k^{n,m}s_k^m-\sigma_k^n\right]^+, \tag{10}$$

which is the same solution as with traditional water-filling.

Define $\bar{\alpha}_k^n$ as the equivalent crosstalk channel gain from user n to the reference line, i.e., $\bar{\alpha}_k^n=\tilde{\alpha}_k^n\mathbf{1}_{\{k\in\bar{\mathcal{K}}\}}$, where $\mathbf{1}_{\{\bullet\}}$ is the indication function. The PSD found with ASB-II is then:

$$s_k^{n,II}\left(w^n,\lambda^n,s_k^{-n}\right)=\left[\frac{w^n}{\lambda^n+\bar{\alpha}_k^n/\tilde{\sigma}_k}-\sum_{m\neq n}\alpha_k^{n,m}s_k^m-\sigma_k^n\right]^+, \tag{11}$$

This is essentially a water-filling type solution, with different water-filling levels for different tones. For this reason we term this algorithm *frequency selective waterfilling*.

V. Convergence Analysis

In this section we prove convergence for both ASB-I and ASB-II in rate adaptive (RA) mode where users fix their weights w^n and aim at maximizing their rates under a total power constraint [6].[3]

A. Convergence in the two-user case

Theorem 1: The ASB-I algorithm converges in a two-user case under fixed $\boldsymbol{w}$ and $\boldsymbol{\lambda}$, if users start from initial PSD values $\left(s_k^1,s_k^2\right)=\left(0,P^2\right)$ or $\left(s_k^1,s_k^2\right)=\left(P^1,0\right)$ on all tones.

The proof of Theorem 1 uses supermodular game theory [7] and strategy transformation similar to [8], and is skipped due to space limitation.

[2] It is different from the conventional water-filling in that here the power level in each tone is not only determined by the dual variables w^n and λ^n, but also by the parameters of the reference line, $\tilde{\alpha}_k^n/\tilde{\sigma}_k$. So each line attempts not only to avoid frequencies with a large amount of noise or crosstalk, but also to avoid frequencies where they will do much damage to the reference line.

[3] The second main category of the spectrum balancing problem is *Fixed Margin* (FM) mode, where users try to minimize their power consumption under a minimum target rate constraint. For example, problem (2) is a mixed RA/FM problem.

Now consider the ASB-II algorithm where two users sequentially optimize their PSD levels under fixed values of $\boldsymbol{w}$, but adjust $\boldsymbol{\lambda}$ to enforce the power constraint at each iteration. The following lemma will be useful in proving the main convergence results.

Lemma 2: Consider any non-decreasing function $f(x)$ and non-increasing function $g(x)$. If there exists a unique x^* such that $f(x^*) = g(x^*)$, and the functions $f(x)$ and $g(x)$ are strictly increasing and strictly decreasing at $x = x^*$ respectively, then $x^* = \arg\min_x \{\max\{f(x), g(x)\}\}$.

Proof of Lemma 2 is given in Appendix A. Denote $s_k^{n,t}$ as the PSD of user n on tone k after iteration t, where $\sum_k s_k^{n,t} = P^n$ is satisfied at the end of any iteration t for any user n. One iteration is defined as one round of updates of all users. We can show that

Theorem 3: The ASB-II algorithm globally converges to the unique fixed point in a two-user system under fixed $\boldsymbol{w}$, if $\max_k \alpha_k^{2,1} \max_k \alpha_k^{1,2} < 1$.

The proof of Theorem 3 is given in Appendix B. The convergence result of iterative water-filling in the two user case [1] is a special case of Theorem 3 (by setting $\tilde{s}_k = 0, \forall k$).

B. Convergence in the N-user case

We further extend the convergence results to a system with an arbitrary $N > 2$ of users. We consider both sequential and parallel PSD updates of the users. In the more realistic but harder-to-analyze parallel updates, time is divided into slots, and the users update their PSDs simultaneously in each time slot according to (11) based on the PSDs from the previous time slot, where the λ^n is adjusted such that the power constraint is tight. The proof of the following theorem is outlined in Appendix B. [4]

Theorem 4: If $\max_{n,m,k} \alpha_k^{n,m} < \frac{1}{N-1}$, then the ASB-II algorithm globally converges (to the unique fixed point) in an N-user system under fixed $\boldsymbol{w}$, with either sequential or parallel updates.

VI. Simulation Results

Here we summarize a typical numerical example comparing the performance of the ASB algorithms with IW, OSB and ISB. A 4 user mixed CO/RT scenario has been selected to make a comparison with the highly complex OSB algorithm possible. As depicted in Fig. 1, user 1 is CO distributed, whilst the other three users are RT distributed. Due to the different distances among the corresponding transmitters and receivers, the RT lines generate strong interferences into the CO line, whilst experiencing very little crosstalk from the CO line. The target rates of users 2 and 3 have both been set to 2 Mbps. For a variety of different target rates of user 4, user 1 (the CO line) attempts to maximize its own data-rate either by transmitting at full power in IW, or by setting its corresponding weight w_{co} to unity in OSB, ISB and ASB. This produces the rate regions shown in Fig 2, which shows that ASB achieves near optimal performance similar as OSB and ISB, and significant gains over IW. For example, with a target rate of 1 Mbps on user 1, the rate on user 4 reaches 7.3 Mbps under ASB algorithm, which is a 121% increase compared with the 3.3 Mbps achieved by IW. We have also performed extensive simulations (more than $10,000$ scenarios) with different CO and RT positions, line lengths and reference line parameters. We found that the performance of ASB is very insensitive to definition of reference line: with a single choice of the reference line we observe good performance in a broad range of scenarios, and consistently significant gains over IW.

[4] This theorem includes the convergence of iterative water-filling in an N-user case with sequential updates (proved in [5]) as a special case. Moreover, the convergence proof for the parallel updates turns out to be simpler than that for sequential updates.

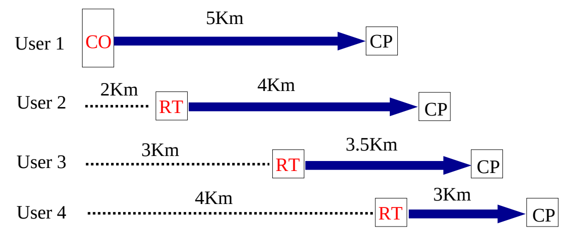

Fig. 1. An example of mixed CO/RT deployment topology.

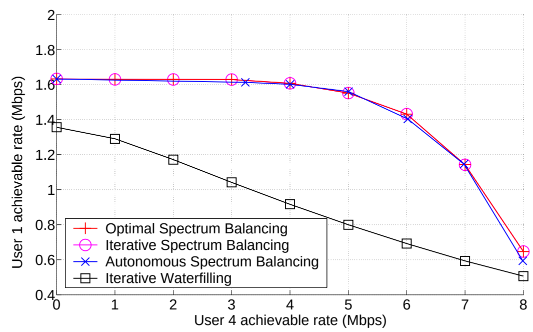

Fig. 2. Rate regions obtained by ASB, IW, OSB, and ISB.

VII. Conclusions

This paper presents the Autonomous Spectrum Balancing (ASB) algorithm, a low complexity, completely autonomous, and close-to-optimal performance method for spectral optimization in frequency selective interference channels. To tackle this nonconvex and coupled optimization problem, ASB uses the concept of a reference line to allow each user to optimize its transmit spectra to achieves its own target rate whilst minimizing the degradation caused to other users within the network. In a DSL context, ASB is shown to achieve near-optimal performance and operates close to the rate-region boundary. Since ASB is completely autonomous, it can be applied in existing DSL modems without the need for a centralized network management center. The convergence of ASB is proven for an arbitrary number of users and under channel conditions that are typically satisfied in a DSL context.

Appendix

A. Proof of Lemma 2

For any $\Delta x > 0$, $f(x^* + \Delta x) > f(x^*) = g(x^*) > g(x^* + \Delta x)$. Similarly for any $\Delta x < 0$, $f(x^* + \Delta x) < f(x^*) = g(x^*) < g(x^* + \Delta x)$. It then can be verified that

$$x^* = \arg\min_x \{\max\{f(x), g(x)\}\}.$$

■

B. Proof of Theorems 3 and 4

We first prove Theorem 3. For any iteration $t \geq 1$ and tone k, user n's PSD is updated as

$$s_k^{n,t+1} = \left[\frac{w^n}{\lambda^{n,t+1} + \bar{\alpha}_k^n/\tilde{\sigma}_k} - \alpha_k^{n,m} s_k^{m,t} - \sigma_k^n\right]^+, \quad (12)$$

where $n, m \in \{1, 2\}$ and $m \neq n$. Here $\lambda^{n,t+1}$ is chosen such that $\sum_k s_k^{n,t+1} = P^n$. Users' PSD are chosen arbitrarily at $t = 0$.

Define $[x]^+ = \max(x, 0)$ and $[x]^- = \max(-x, 0)$, then for any user n we have

$$\sum_k \left[s_k^{n,t} - s_k^{n,t'}\right]^+ = \sum_k \left[s_k^{n,t} - s_k^{n,t'}\right]^-, \forall t, t' \geq 1, \quad (13)$$

since the total power constraint is always satisfied at the end of any iteration. Also define

$$f^{n,t}(x) = \sum_k \left[\left[\frac{w^n}{x + \bar{\alpha}_k^n/\tilde{\sigma}_k} - \alpha_k^m s_k^{m,t} - \sigma_k^n\right]^+ - s_k^{n,t}\right]^-,$$

$$g^{n,t}(x) = \sum_k \left[\left[\frac{w^n}{x + \bar{\alpha}_k^n/\tilde{\sigma}_k} - \alpha_k^m s_k^{m,t} - \sigma_k^n\right]^+ - s_k^{n,t}\right]^+.$$

It is clear that $f^{n,t}(x)$ ($g^{n,t}(x)$, respectively) is non-decreasing (non-increasing) in x, and strictly increasing (strictly decreasing) at $x = \lambda^{n,t+1}$ if λ has not converged.

Now we can show that

$$\max\left\{\sum_k \left[s_k^{1,t+1} - s_k^{1,t}\right]^+, \sum_k \left[s_k^{1,t+1} - s_k^{1,t}\right]^-\right\} \quad (14)$$

$$= \max\left\{f^{1,t}\left(\lambda^{1,t+1}\right), g^{1,t}\left(\lambda^{1,t+1}\right)\right\} \quad (15)$$

$$\leq \max\left\{f^{1,t}\left(\lambda^{1,t}\right), g^{1,t}\left(\lambda^{1,t}\right)\right\} \quad (16)$$

$$\leq \max\left\{\sum_k \left[\alpha_k^{1,2}(s_k^{2,t-1} - s_k^{2,t})\right]^+, \sum_k \left[\alpha_k^{1,2}(s_k^{2,t-1} - s_k^{2,t})\right]^-\right\} \quad (17)$$

$$\leq \max\left\{\sum_k \alpha_k^{1,2}\left[s_k^{2,t-1} - s_k^{2,t}\right]^+, \sum_k \alpha_k^{1,2}\left[s_k^{2,t-1} - s_k^{2,t}\right]^-\right\} \quad (18)$$

$$\leq \max_k \alpha_k^{1,2} \max\left\{\sum_k \left[s_k^{2,t} - s_k^{2,t-1}\right]^+, \sum_k \left[s_k^{2,t} - s_k^{2,t-1}\right]^-\right\} \quad (19)$$

$$\leq \max_k \alpha_k^{1,2} \max_k \alpha_k^{2,1} \cdot \max\left\{\sum_k \left[s_k^{1,t} - s_k^{1,t-1}\right]^+, \sum_k \left[s_k^{1,t} - s_k^{1,t-1}\right]^-\right\} \quad (20)$$

$$< \max\left\{\sum_k \left[s_k^{1,t} - s_k^{1,t-1}\right]^+, \sum_k \left[s_k^{1,t} - s_k^{1,t-1}\right]^-\right\} \quad (21)$$

where (15) follows from the definition of $f^{n,t}$ and $g^{n,t}$, (16) follows from Lemma 2 and letting $x = \lambda^{n,t}$, (17) follows from the definition of $f^{n,t}$ and $g^{n,t}$, the expression of $s_k^{1,t}$ in (12), and the fact that $[x^+ - y^+]^+ \leq [x - y]^+$ and $[x^+ - y^+]^- \leq [x - y]^-$ for any x and y, (18) follows by using $\sum_k [x_k y_k]^+ \leq \sum_k x_k [y_k]^+$ and $\sum_k [x_k y_k]^- \leq \sum_k x_k [y_k]^-$ for any $x_k \geq 0$ and y_k, (20) is obtained by applying the arguments from (15) to (19) again, and finally (21) follows from the condition in Theorem 3. This shows that the ASB-II algorithm is a contraction mapping from any initial PSD values, thus globally converges to a unique fixed point [9, Page 183].

Next we prove Theorem 4. First, the convergence of sequential updates can be proved using similar techniques as in [5] and thus is omitted here. Now in the more realistic parallel updates case, the PSD of user n in tone k after iteration $t+1$ is

$$s_k^{n,t+1} = \left[\frac{w^n}{\lambda^{n,t+1} + \bar{\alpha}_k^n/\tilde{\sigma}_k} - \sum_{m \neq n} \alpha_k^{n,m} s_k^{m,t} - \sigma_k^n\right]^+.$$

The rest of the proof can be obtained similar as for Theorem 3 by replacing (14) to (21) with the following derivations:

$$\max_n \max\left\{\sum_k \left[s_k^{n,t+1} - s_k^{n,t}\right]^+, \sum_k \left[s_k^{n,t+1} - s_k^{n,t}\right]^-\right\}$$

$$\leq \max_n \max\left\{\sum_k \left[\sum_{m \neq n} \alpha_k^{n,m}\left(s_k^{m,t} - s_k^{m,t-1}\right)\right]^+, \sum_k \left[\sum_{m \neq n} \alpha_k^{n,m}\left(s_k^{m,t} - s_k^{m,t-1}\right)\right]^-\right\}$$

$$\leq \max_n \max\left\{\sum_k \left[(N-1)\max_{m \neq n} \alpha_k^{n,m}\left(s_k^{m,t} - s_k^{m,t-1}\right)\right]^+, \sum_k \left[(N-1)\min_{m \neq n} \alpha_k^{n,m}\left(s_k^{m,t} - s_k^{m,t-1}\right)\right]^-\right\}$$

$$= (N-1)\max_n \max_{m \neq n} \max\left\{\sum_k \left[\alpha_k^{n,m}\left(s_k^{m,t} - s_k^{m,t-1}\right)\right]^+, \sum_k \left[\alpha_k^{n,m}\left(s_k^{m,t} - s_k^{m,t-1}\right)\right]^-\right\}$$

$$\leq (N-1)\max_{n,m,k} \alpha_k^{n,m} \max_n \cdot \max\left\{\sum_k \left[s_k^{m,t} - s_k^{m,t-1}\right]^+, \sum_k \left[s_k^{m,t} - s_k^{m,t-1}\right]^-\right\}$$

$$< \max_n \max\left\{\sum_k \left[s_k^{m,t} - s_k^{m,t-1}\right]^+, \sum_k \left[s_k^{m,t} - s_k^{m,t-1}\right]^-\right\}.$$

The last inequality is due to the condition that $\max_{n,m,k} \alpha_k^{n,m} < \frac{1}{N-1}$. By similar arguments as in Theorem 3, the algorithm globally converges to a unique fixed point. ■

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