Estimate the Classical Information Rate of a Quantum Channel with Memory

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Classical Communication over a Quantum Channel with Memory

We consider the problem of transmitting classical information over a time-invariant quantum channel with memory. In particular, we consider quantum channels as

$$N : \mathcal{D}(\mathcal{H}^A \otimes \mathcal{H}^B) \rightarrow \mathcal{D}(\mathcal{H}^A \otimes \mathcal{H}^B),$$

(1)

with input/output quantum system $A, B$, hidden memory system $S$, $\hat{S}$ (before and after each channel use). The classical communication over such a channel can be accomplished by mapping the classical source $X$ to the input system $A$:

$$x \mapsto \rho_x^A, \quad \forall x \in X,$$

(2)

and then applying the measurement $\{M_y^B\}_{y \in Y}$ on the output system $B$. Such a process imposes the following two channel rules:

**Output Rule** given previous state $\rho^{S_{t-1}}$:

$$P_{\rho|\rho^{S_{t-1}}}(y_t = y, x_t = x) = tr \left( (M_y^B \otimes I) N \left( \rho_x^A \otimes \rho^{S_{t-1}} \right) (M_y^B \otimes I) \right).$$

(3)

**Transition Rule** given current channel output $y_t$:

$$\rho^{S_{t+1}} = tr_B \left( (M_y^B \otimes I) N \left( \rho_x^A \otimes \rho^{S_{t}} \right) (M_y^B \otimes I) \right).$$

(4)

Estimating the Information Rate (IR)

Given a quantum-state channel we are interested in finding its information rate with given input distribution $Q(x)$, namely:

$$I(Q, E^{(n)}) = \lim_{n \to \infty} \frac{1}{n} \left( H(Y_1, Y_2, \ldots, Y_n) - \log \sum_{y \in Y} \left| \langle y | Y \rangle \right|^2 \right),$$

(5)

where $Q = \{q_y(y)\}_{y \in Y}$ is the input distribution. To estimate the above three terms by $(1/\log \#Y, 1/\log \#Y, 1/\log \#Y)$, respectively, an example following algorithm estimates $\frac{1}{n} H(Y_1, \ldots, Y_n)$.

**Algorithm 1: Estimation of $1/n H(Y_1, \ldots, Y_n)$**

1. **Require:** Quantum-State Channel $\{E^{(n)}\}_{n \in \mathbb{Z}^+}$ PMF $Q(x)$, and $n \in \mathbb{Z}^+$
2. **Ensure:** $H(Y_1, \ldots, Y_n) \approx \sum_{y \in Y} \log \lambda_y$
3. Generate a channel output $y_1, \ldots, y_n$ using the original channel;
4. Initialize $\sigma_1 \leftarrow \rho^{Y}$;
5. for each $\ell = 1, 2, \ldots, n$ do
6. $\sigma_\ell \leftarrow \sum_{y \in Y} Q(x_\ell) \cdot E^{(1)} (\sigma_{\ell-1}, y)$;
7. $\lambda_\ell \leftarrow tr (\sigma_\ell)$;
8. $\sigma_\ell \leftarrow \sigma_\ell / \lambda_\ell$;
9. end for

Example: Quantum Gilbert–Elliott channel

The Quantum Gilbert–Elliott channel is a quantum channel on a 2-qubit system:

$$X^{GE}[x] : \{x\} \otimes \{y\} \rightarrow \{x\} \otimes \{y\} \in \{X, Y, E_1, E_2, E_3\},$$

(7)

where $\{X, Y, E_1, E_2, E_3\}$ are two quantum bit-flip channel with 'flipping rate' $p_x$ and $p_y$, respectively. Here, $|s\rangle$ is regarded as the hidden memory qubit, and $|a\rangle$ is the input qubit. For classical communication, we map the binary source $x \in \{0, 1\}$ as $x \mapsto |x\rangle$, and apply projective measurement $w_r$ to $y$. We also assume a unitary evolution $U = \exp(-\text{i}a H)$ occurring on the memory qubit between each two channel use:

\[ U_{\rho} |s\rangle \langle s| = \begin{bmatrix} 1 & 0 \\ 0 & \rho \end{bmatrix}. \]

Quantum-State Channels

Notice that any classical communication scheme over a quantum channel with memory can be fully specified by the class of mappings $E^{(t)} : \mathcal{H}^A \rightarrow \mathcal{H}^B$, where

- $X, Y$ are the input and output alphabet, respectively;
- $E^{(t)} : \mathcal{H}^A \rightarrow \mathcal{H}^B$ is some quantum operation for each $x \in X$ and $y \in Y$;
- $S$ and $\hat{S}$ represent the state quantum systems before and after the channel use;
- for all $x \in X$, $y \in Y$: $\sum_{y \in Y} E^{(t)}(\rho \otimes \rho^{S_{t-1}}) \leq 1$, i.e., $\sum_{y \in Y} E^{(t)}(\rho)$ is trace preserving;
- Given previous state $\rho$. $P_{xy}(y, x) = tr(E^{(t)}(\rho))$;
- Given current channel output being $y$, next state $\rho \propto E^{(t)}(\rho)$.

A Graphical Notation

Quantum-State channels can be described by the factor graph on the right-hand side. For the given operator-sum representative

$$E^{(t)}(\rho) = \sum_{y \in Y} E_{xy}(\rho)\rho^{S_{t-1}}$$

(9)

the function $W$ on the RHS can be written as

$$W^{(r)}(s_r, s_{r-1}; s_{r+1}, s_{r+2}) = \sum_{y \in Y} E_{xy}(s_r, s_{r-1}) E_{xy}(s_{r+1}, s_{r+2})$$

(10)

Bounding IR with Auxiliary Channel

The complexity of Algorithm 1 is linear w.r.t. $n$, but exponential w.r.t. the dimension of the involved quantum channel. A workaround is to consider simpler auxiliary channels, which yields upper and lower bounds on the information rate. For any quantum-state channel $\{E^{(t)}\}_{t \in \mathbb{Z}}$, define the induced channel rule $W_{E^{(t)}}$ and induced output distribution $QW_{E^{(t)}}$ as

$$W_{E^{(t)}}(y) = \sum_{x \in X} E^{(t)}(\rho \otimes \rho^{S_{t-1}}) \log W_{E^{(t)}}(y, \rho).$$

(11)

and lower bound

$$L^{(n)}(s_1) = \frac{1}{n} \sum_{y \in Y} Q(x, y) W_{E^{(t)}}(y, \rho) \log W_{E^{(t)}}(y, \rho).$$

(12)

There are two important remarks:
1. Expression (13) and (14) can be estimated in the same fashion as in Algorithm 1.
2. One interesting problem is to find optimal $\hat{E}$ minimizing

$$\Delta(\hat{E}) = \frac{1}{n} \sum_{y \in Y} Q(x, y) W_{E^{(t)}}(y, \rho) \log W_{E^{(t)}}(y, \rho).$$

(15)

(We know how to solve this problem in this classical case.)